Seminar On Fukaya Categories

typeset by walker h. stern

The following notes were taken from a seminar given at Universität Bonn during the winter semester 2015-2016, overseen by Prof. Dr. Tobias Dyckerhoff and Prof. Dr. Catharina Stroppel.

Contents

Motivation

Tobias Dyckerhoff

The general outline of the seminar will be as follows:

I A_∞-categories – 3 talks

II Fukaya categories – 3 talks

III Seidel-Fukaya categories – 3 talks

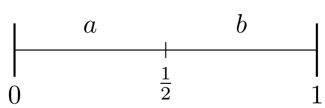
IV Additional topics – 4 talks

- Khovanov's tangle/knot invariants
- Fukaya Categories of surfaces and staility conditions

In this introductory talk, we will attempt to provide basic motivation for the first three sections.

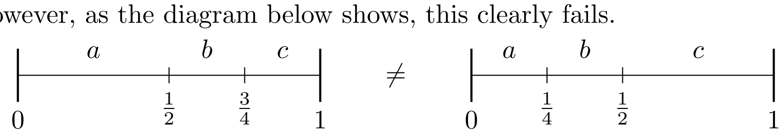
A_∞ *Spaces*

For a pointed topological space $(X, *)$, we can form the loop space $Y := \mathrm{Map}\big((S^1, *), (X, *)\big)$. [1] We can then form a composition operation, as in the construction of the fundamental group[2].

$m(a, b) \quad =$ 

this composition law can be represented as an element $m_2 \in \mathrm{Map}(Y^2, Y)$.

Ideally, m_2 would satisfy the most basic of properties: associativity. However, as the diagram below shows, this clearly fails.



This is where quotienting by the equivalence relation defined by homotopy would give associativity, however, if we don't want to lose information we can instead choose a homotopy

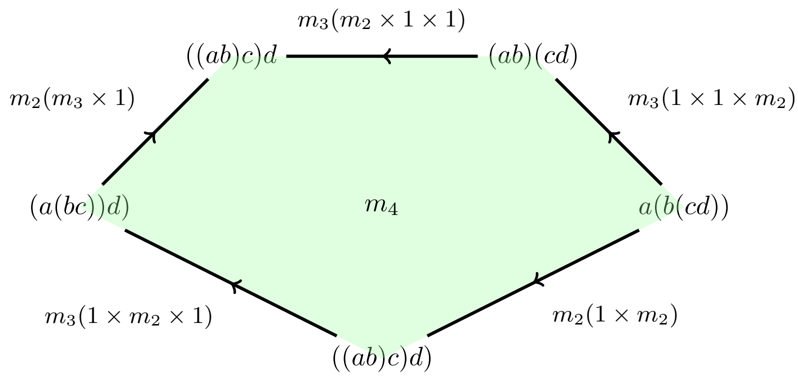
$$m_2(a, m_2(b, c)) \xrightarrow{m_3} m_2(m_2(a, b), c)$$

[1] The general theory discussed in this section is due to Stasheff.
[2] Notice that in the normal construction of the fundamental group, we would immediately pass to homotopy classes of loops, rather than considering the full space Y.

which can be seen as a map $m_3 : I \to \mathrm{Map}(Y^3, Y)$, such that

$$\partial(m_3) = m_2(m_2 \times 1) - m_2(1 \times m_2)$$

If we begin with four loops, we can look at all possible compositions, and repeat this construction, ie, for $a, b, c, d \in Y$, we get



The boundary of the figure may be viewed as a map $S^1 \to \mathrm{Map}(Y^4, Y)$, and so a choice of m_4 can be seen as a map $m_4 : D^2 \to \mathrm{Map}(Y^4, Y)$, such that the following diagram commutes.[3]

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] & \mathrm{Map}(Y^4, Y) \\ D^2 \arrow[ur, "m_4"'] & \end{tikzcd}$$

[3] As before, we can express this in the form of an equation, ie, $\partial m_4 = m_3(m_2 \times 1 \times 1 - 1 \times m_2 \times 1 + 1 \times 1 \times m_2) - m_2(m_3 \times 1 + 1 \times m_3)$

Continuing this process, we find that the high coherence conditions m_i are defined by maps of balls (more precisely, of convex polytopes, the so called 'Stasheff Polytopes') into $\mathrm{Map}(Y^i, Y)$.

Definition. An A_∞*-space* is a topological space Y equipped with operations

$$m_n : K_n \to \mathrm{Map}(Y^n, Y)$$

satisfying the equations

$$\begin{aligned} \partial(m_3) &= m_2(m_2 \times 1) - m_2(1 \times m_2) \\ \partial(m_4) &= m_3(m_2 \times 1 \times 1 - 1 \times m_2 \times 1 + 1 \times 1 \times m_2) \\ &\quad - m_2(m_3 \times 1 + 1 \times m_3) \\ \vdots \; &\;\vdots \quad \vdots \end{aligned}$$

We have the example, of course, of $Y = \Omega_* X$, but the following theorem tells us in some sense that this is the only example.

Theorem (Stasheff)**.** *Let Y be an A_∞-space such that $(\pi_0(Y), m_2)$ is a group, then there exists a pointed space $(X, *)$ and a homotopy equivalence $Y \sim \Omega_* X$ as A_∞-spaces.*

This construction can be generalized. A linearized version is called an A_∞-algebra and a multi=object version is called an A_∞-category.

Fukaya Categories

We now begin from the perspective of Morse Theory. Let X be a smooth n-dimensional closed manifold, and consider $f : X \to \mathbb{R}$ a morse function[4].

[4] That is, a smooth function with only non-degenerate critical points, whose set is denoted by $\mathrm{crit}(f)$.

Locally around $p \in \mathrm{crit}(f)$, we can express f as

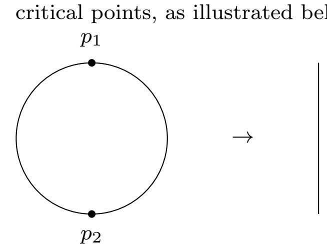
$$f = x_1^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_n^2$$

The number of negative signs appearing in that expression depends only on p, and we call it $\mathrm{index}(p)$.

From Morse Theory, we know that there exists a CW-structure on X with[5]

$$\left\{ \substack{\text{cells of} \\ \text{dimension } k} \right\} \cong \left\{ \substack{\text{critical points} \\ \text{of index } k} \right\}$$

[5] For example, we can take the projection $\pi : S^1 \to \mathbb{R}$. This has two critical points, as illustrated below.



p_1 has index 0, and p_2 has index 1, so we expect to find a cell structure for S^1 containing a single 0-cell and a single 1-cell. However, this is precisely the standard cell structure for the circle.

This tells us that there exists a chain complex freely generated by critical points which computes $H_*(X; \mathbb{R})$. In turn, this implies that

$$\#\,(\text{crit points of dim} k) \geq b_k(X)$$

where b_k is the kth betti number.

Now, we want to tie this example to the formalism of sympectic geometry, whence Fukaya categories arise. We do this loosely following a maxim (attributed to Weinstein).

The Lagrangian Creed: Everything is a Lagrangian Submanifold.

Example. For a manifold X as above, the cotangent bundle T^*X is a symplectic manifold with form ω given locally by

$$\sum_{i=1}^{n} dx_i \wedge d\xi_i$$

- The zero section $L_0 := X \subset T^*X$ is a lagrangian submanifold.
- for f a smooth function on X, $df \in \Gamma(T^*X)$ ie df is a section

$$df : X \to T^*X$$

The image $L_1 := \mathrm{im}(df) \subset T^*X$ is a lagrangian submanifold.

We also have $L_0 \cap L_1 = \mathrm{crit}(f)$, and that these intersections are transverse when the critical points are non-degenerate.

Having moved to the formlism of symplectic geometry, we have the following 'theorem'

Theorem (Floer). *Let L_1 and L_0 be compact transverse Lagrangians in a symplectic manifold (M, ω) (+ various technical conditions), then there exists a natural complex*

$$CF(L_0, L_1)$$

generated freely by $L_0 \cap L_1$ such that

1. *$H_*(CF(L_0, L_1))$ is invariant under hamiltonian isotopies[6].*
2. *If L_1 is Hamiltonian isotopic to L_0, then*

$$H_*(CF(L_0, L_1)) \cong H_*(L_0)$$

[6] Recall that, given a smooth function (the 'Hamiltonian function') $F : M \to \mathbb{R}$, we get a differential equation

$$\omega(X_f, -) = df$$

whose unique solution X_f is refered to as the *Hamiltonian vector field* of f. Then, we can define the *Hamiltonian isotopy* (or *Hamiltonian flow*) of f to be ϕ_t^f, the unique flow along X_f.

And finally, we have

Theorem (Fukaya). *There exists an A_∞-category* $\mathrm{Fuk}(M\omega)$ *with objects given by Lagrangian submanifolds + technical decorations, and morphism complexes $CF(L_0, L_1)$*

Seidel-Fukaya Categories

Let T be a triangulated category subject to the requirements that $\mathrm{Hom}^*(E, F)$ be finite dimensional for all objects E, F of T, and that T be linear over $\overline{k} = k$.

Definition. A collection of objects $(E_1, E_2, \ldots E_n)$ in T is called *exceptional* if

1. $\mathrm{Hom}^*(E_i, E_i) = k \cdot id$ for every i
2. $\mathrm{Hom}^*(E_i, E_j) = 0$ for every $i > j$

It is called *full* if

$$\langle E_1, \cdots E_n \rangle = T$$

On such expectionally collections, we have the operation of *Mutation*, which preserves exceptional-ness. For any $1 \leq i < n$ mutation is given by

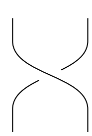
$$(E_1, E_2, \ldots E_i, E_{i+1}, \ldots E_n) \mapsto (E_1, \ldots T_{E_i}(E_{i+1}), E_i, \ldots E_n)$$

Where T_{E_i} is given by

$$T_{E_i}(E_{i+1}) = \mathrm{cone}\left(\mathrm{Hom}^*(E_i, E_{i+1}) \otimes E_i \to E_{i+1}\right)$$

That is, the cone over the evaluation map.

By sending the relevant twist



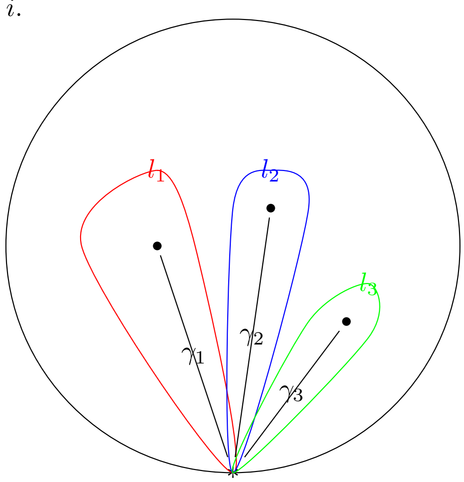
to the mutation on $i, i+1$, we find that the set of exceptional collections of length n comes equipped with an action of the Artin Braid Group.

Example.

$$\begin{aligned} T = D^b(A_n\text{-mod}) \quad &= \quad \langle E_1 \to E_2 \to \cdots \to E_n \rangle \\ &\qquad\qquad\qquad \downarrow \\ &\quad\ \ \langle E_1 \leftarrow T_{E_1}(E_2) \to E_3 \to \cdots \to E_n \rangle \end{aligned}$$

Applying once again the Lagrangian Creed, we translate this into '0-dimensional symplectic geometry'

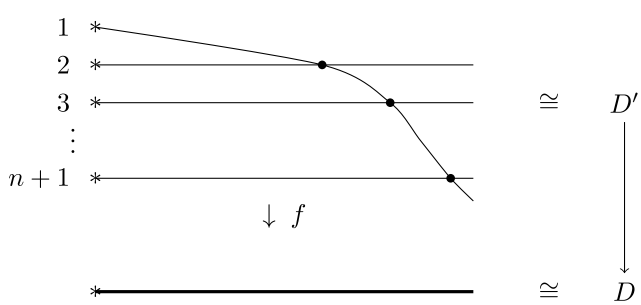
We start with a pointed disk with n marked points M, and choose loops l_i represenatives of $\pi_1(D \setminus M)$ coresponding to curves γ_i from $*$ to i.



From here, we can construct a ramified cover of degree $n+1$ by specifying monodromy:

$$l_i \mapsto (1, i) \in S_{n+1} \curvearrowright f^{-1}(*)$$

Schematically, we get something like:



and we have that $f^{-1}(*)$ is a 0-dimensional symplectic manifold, with

$$\{1,2\}, \{1,3\}, \ldots \{1, n+1\} \subset f^{-1}(*)$$

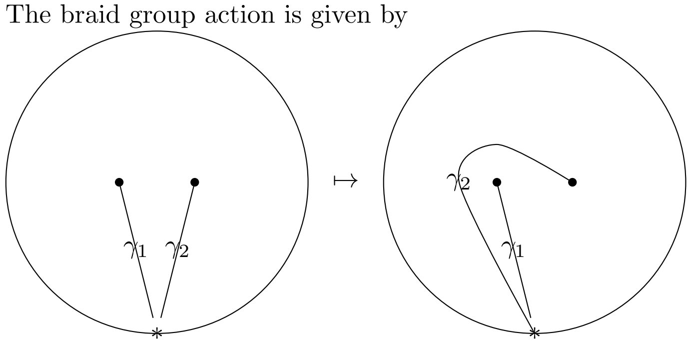
lagrangian 0-spheres which we will call *vanishing cycles*.

Theorem (Seidel)**.**

$$\langle \{1,2\}, \{1,3\}, \dots \{1, n+1\} \rangle = \mathrm{Fuk}(f) \cong D^b(A_n\textit{-mod})$$

and $\{1,2\}, \{1,3\}, \dots \{1, n+1\}$ *is a distinguished basis of vanishing cycles corresponding to* $\{\gamma_i\}$*.*

The braid group action is given by



Seidel's theory generalizes this: It claims that this works for any (reasonable) symplectic lefschetz fibration $f : X \to D$.

Part I

A_∞ Categories

A_∞ *Spaces*

Michael Brown

The rough goal of this talk is to explain and provide a proof for the theorem of Stasheff:

Theorem (Stasheff)**.** *Let X be a (pointed) space. Then X is a loop space if and only if X is an A_∞-space and $\pi_0(X)$ is a group under the monoid structure induced by the A_∞-structure.*

Before we begin, we fix some notation[7]:

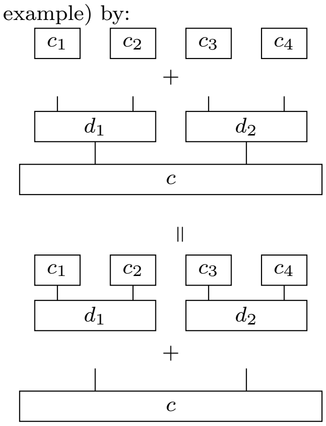
$$\begin{array}{rcl} \mathcal{U} & := & \text{Category of compactly generated} \\ & & \text{Hausdorff Spaces} \\ \mathcal{T} & := & \text{Category of spaces in } \mathcal{U} \\ & & \text{with non-degenerate base point} \end{array}$$

[7] Here 'non-degenerate base point' means that the inclusion $* \to X$ is a cofibration. More precisely, using the standard model structure, X is a retract of a CW complex.

Definition. An *operad* $\mathcal{C}$ consists of spaces $\mathcal{C}(j) \in \mathcal{U}$ for every $j \geq 0$, where $\mathcal{C}(0) = *$, along with the data:

1. Continuous maps
$$\gamma : \mathcal{C}(k) \times \mathcal{C}(j_1) \times \cdots \times \mathcal{C}(j_k) \to \mathcal{C}(j)$$
where $j = \sum j_i$, such that the following condition holds:
Given $c \in \mathcal{C}(k)$, $d_s \in \mathcal{C}(j_s)$, and $c_t \in \mathcal{C}(j_t)$,
$$\begin{array}{rcl} \gamma(\gamma(c; d_1, \ldots, d_k); c_1, \ldots, c_j) & = & \gamma(c; f_1, \ldots, f_k) \quad \text{where} \\ f_s & = & \gamma(d_s; c_{j_1+\cdots+j_{s-1}+1}, \ldots, c_{j_1+\cdots+j_s}) \end{array}$$
and where $f_s = *$ if $j_s = 0$.[8]
2. An identity element $1 \in \mathcal{C}(1)$ st $\gamma(1; x) = x$ for all $x \in \mathcal{C}(k)$ and for any k, and $\gamma(c; 1, \ldots, 1) = c$ for all $c \in \mathcal{C}(k)$ and for any k.
3. A right action of the symmetric group S_n on $\mathcal{C}(k)$ for every $k \geq 0$ such that for $c \in \mathcal{C}(k)$, $d_s \in \mathcal{C}(j_s)$, $\sigma \in S_k$ and $\tau_s \in S_{j_s}$:
   i)
   $$\gamma(c\sigma; d_1, \ldots, d_k) = \gamma(c; d_{\sigma^{-1}(1)}, \ldots, d_{\sigma^{-1}(k)})$$
   where $\sigma(j_1, \ldots, j_k) \in S_j$ permutes the k blocks of letters determined by the partition $j_1, \ldots, j_k$ of j in the same way that σ permutes k letters.

[8] An operad can be more productively thought of as a set of 'black boxes' representing n-ary operations. In this sense, the condition 1. can be diagrammatically represented (in an example) by:



ii) $\gamma(c; d_1\tau_1, \ldots, d_k\tau_k) = \gamma(c; d_1, \ldots, d_k)(\tau_1 \times \cdots \times \tau_k)$

Definition. A *morphism* $\mathcal{C} \to \mathcal{C}'$ of operads is a collection of θ_j : $\mathcal{C}(j) \to \mathcal{C}'(j)$ sending $1 \in \mathcal{C}(1)$ to $1 \in \mathcal{C}'(1)$, S_k equivariant for all k, and compatible with γ maps.

Example. Let $(X, *) \in \mathcal{T}$. The endomorphism operad $\mathcal{E}_X$ is given by:[9]

$$\mathcal{E}_X(j) = \mathrm{Map}(X^j, X)$$

with γ maps defined by

$$\gamma(f; g_1, \ldots, g_k) = f \circ (g_1 \times \cdots \times g_k)$$

the identity element is $id_X \in \mathcal{E}_X(1)$ and the actions permute the components of the product, i.e. for $\sigma \in S_k$ and $g \in \mathcal{E}_X$

$$(g\sigma)(x_1, \ldots, x_k) = g(x_{\sigma-1(1)}, \ldots, x_{\sigma^{-1}(k)})$$

Fact. $\mathcal{E}_X$ is an operad.

Definition. An *action of an operad* $\mathcal{C}$ on a space X is a morphism

$$\mathcal{C} \to \mathcal{E}_X$$

We can use this terminology to rephrase the Stasheff theorem from the beginning.

Theorem (May, 1972 (Boardman-Vogt, 1973))**.** *There exist operads $\mathcal{C}_n$ for all $1 \leq n \leq \infty$ such that if X is path connected, then X is a $\mathcal{C}_n$ space if and only if X is an n-fold loop space.*

Definition. An operad $\mathcal{C}$ is an A_∞ *operad* if $\pi_0(\mathcal{C}(j)) \cong S_j$[10] and each path component of $\mathcal{C}(j)$ is contractible[11]

An A_∞ *Space* is a space with an action of an A_∞ operad.

Definition. A *Little Interval* is a function

$$c : I \to I$$

of the form $c(t) = (y - x)t + x$ for some $0 \leq x < y \leq 1$.

Definition. The *Little Intervals Operad* $\mathcal{L}$ ($= \mathcal{C}_1$ from the theorem of May), consists of spaces $\mathcal{L}(n)$ of ordered n-tuples $(c_1, \ldots, c_n)$ of little intervals such that the collection $\{c_i(I^\circ)\}_{i=1}^n$ are mutually disjoint. The action of S_n on $\mathcal{L}(n)$ permutes the little intervals. The identity in $\mathcal{L}(a)$ is id_I, and the map

$$\gamma : \mathcal{L}(k) \times \mathcal{L}(j_1) \times \cdots \times \mathcal{L}(j_k) \to \mathcal{L}(j)$$

takes collections $(c_1, \ldots, c_k) \in \mathcal{L}(k)$ and $(b_1^i, \ldots, b_{j_i}^i) \in \mathcal{L}(j_i)$ and sends them to the collection

$$(c_1 \circ b_1^1, c_1 \circ b_2^1, \ldots, c_1 \circ b_{j_1}^1, \ldots, c_k \circ b_1^k, \ldots, c_k \circ b_{j_k}^k) \in \mathcal{L}(j)$$

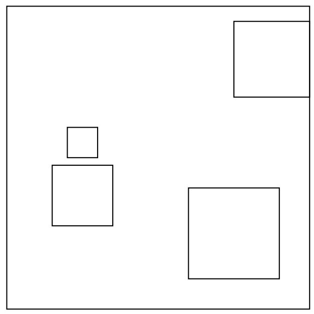
[9] $\mathrm{Map}(X^j, X)$ is here taken to be the pointed topological space of continuous based maps in the compact-open topology

[10] That is, $\pi_0(\mathcal{C}(j))$ is a free transitive S_j space (a torsor).

[11] Technically, one also needs additional cofibrancy conditions in this definition, since otherwise the associative operad would be considered an A_∞ operad (the associative operad has $\mathcal{C}(j) = S_j$ with the discrete topology). However, our chosen A_∞ operad, the little disks operad, satisfies these additional conditions, and so it is sufficient to work with this example.

This is, in fact, an A_∞ operad. One can then take the definition of an A_∞ space to be a space with an $\mathcal{L}$-action.

Remark. The higher $\mathcal{C}_n$ from May's theorem are called the *little n-cube operads*. Spaces over them are called E_n-spaces. While we won't delve into the construction here (it is very similar to the little disks operad), schematically, we get something like:



We can also retrieve the 'homotopy associativity conditions' from the previous talk from an $\mathcal{L}$ action. Suppose

$$\theta : \mathcal{L} \to \mathcal{E}_X$$

is a morphism. Then we have data $\theta_2 : \mathcal{L}(2) \to \mathrm{Map}(X^2, X)$, so that for any $z \in \mathcal{L}(2)$ we get a 'multiplication' $\theta_2(z)$ on the space X.

If z_1 and z_2 are connected by a path in $\mathcal{L}(2)$, this yields a homotopy from $\theta_2(z_1)$ to $\theta_2(z_2)$. However, we also have that

$$\begin{aligned} \pi_0(\mathcal{L}(2)) &\cong S_2 \\ \theta_2(z.(1\ 2)) &= \theta_2(z).(1\ 2) \\ \theta_2(z.(1\ 2))(x_1, x_2) &= \theta_2(z)(x_2, x_1) \end{aligned}$$

So we can see that, in general, there is no reason to expect our multiplication to be homotopy commutative.

However, for associativity, it works out. For ease of writing, let us take $c \in \mathcal{L}(2)$ that preserves the order of the intervals. We can see that there is a path from $\gamma(c; c, 1)$ to $\gamma(c; 1, c)$ in $\mathcal{L}(3)$[12]. We can also find a homotopy unit: $\gamma(c; 1, *)$ and $\gamma(c; *, 1)$ are both connected by a path to $1 \in \mathcal{L}(1)$.

[12] Both preserve the order, so it isn't hard to show.

Fact. If $Y \in \mathcal{T}$, then ΩY is an $\mathcal{L}$-space.

This is relatively easy to see, using many of the same ideas and constructions as were sketched in the first section of the previous talk. The hard direction to prove in May's theorem is the converse.

Proof (sketch). Given a monad C in $\mathcal{T}$, a functor F over C, and an algebra X over C, one can construct a simplicial space $B_\bullet(F, C, X)$.

The geometric realization of $B_\bullet(F, C, X)$ is called the *bar construction* of the triple (F, C, X) (See chapter 9 of May's book, listed in the references).

It is known that an operad $\mathcal{C}$ induces a monad, and a $\mathcal{C}$-space yields an algebra over this monad[13]. So, suppose that X is path-connected and has an action of $\mathcal{L}$, then we have the induced monad, $L = \Omega S$ the loop space of the suspension, and the space X. So we can form

$$\begin{aligned} X &\overset{HE}{\cong} |B_\bullet(L, L, X)| \\ &\overset{HE}{\cong} |B_\bullet(\Omega S, L, X)| \\ &\overset{HE}{\cong} \Omega\, |B_\bullet(S, L, X)| \end{aligned}$$

□

[13] As an example of this generalized bar construction, we first note that if we think of F and C as endofunctors, then

$$B_n(F, C, X) = FC^n X$$

So, if A is an associative algebra over k, then

$$A \otimes_k - : \mathrm{Mod}(A) \to \mathrm{Mod}(A)$$

this is a monad in $\mathrm{Mod}(A)$, and a module M over A is an algebra over this monad. So, via Dold-Kan, we get

$$B_\bullet(A, A, M) \to \text{Classical Bar Resolution}$$

A_∞ Algebras and A_∞ Categories

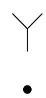
WALKER STERN

The first observation to make in defining A_∞ algebras is that there is a model for the (non-sigma) topological A_∞-operad given by

$$A_\infty^{Top}(n) = K_n$$

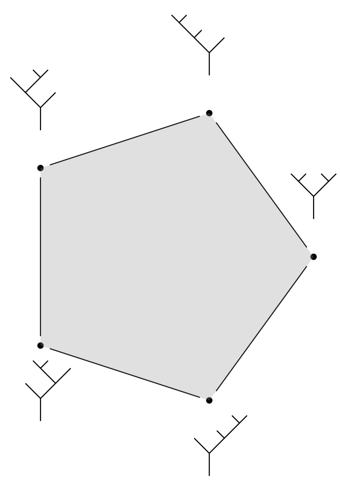
where K_n is the n^th Stasheff Polytope. Importantly, we can give a very natural cell structure on the K_n, with 0-cells representing trees in the following way:

$$A_\infty^{Top}(2) \quad =$$



$$A_\infty^{Top}(3) \quad =$$

$$A_\infty^{Top}(4) \quad =$$



So we see that, in this cell structure, K_n has precisely one $n-2$-cell.

We can then define a (non-unital) non-symmetric operad in chain complexes by

$$A_\infty^{dg}(n) := C_\bullet^{cell}(A_\infty^{Top}(n) \qquad n \geq 2$$

So, for a complex (V, d_V) letting the endomorphism operad[14] be

$$\mathrm{End}_V(n) := {}_{V^{\otimes n} \to V \text{graded by degree}}^{\text{Operad of maps of graded VS's}}$$

[14] The differential on $\mathrm{End}_V(n)$ is given by the formula

$$\partial_V(f) = d_V \circ f - (-1)^{|f|} f \circ d_{V^{\otimes n}}$$

Looking at the first few terms, we notice that each map $A_\infty^{dg}(n) \to \mathrm{End}_V$ amounts to selecting a single map m_n of degree $n-2$, subject to some conditions. So we can define:

Definition. An A_∞-algebra is a Chain complex V with differential m_1 and maps (for $n \geq 2$)

$$m_n : V^n \to V$$

of degree $n-2$ such that

$$\sum_{p+q+r=n} (-1)^{p+qr} m_k \circ (id^{\otimes p} \otimes m_q \otimes id^{\otimes r}) = 0$$

where $k = p+1+r$. Alternately, we can express this condition in terms of the differential ∂_V on End_V:

$$\partial_V(m_n) = - \sum_{\substack{n=p+q+r \\ k>1 \quad q>1}} (-1)^{p+qr} m_k \circ (id^{\otimes p} \otimes m_q \otimes id^{\otimes r})$$

It is worth noting at this juncture, that one can choose the convention that differentials move upwards (cohomological convention) instead of the convention chosen here (homological convention). The main material effect of doing so is that in the cohomological convention, m_n is of degree $2-n$.

Example. The singular chain complex $C_\bullet^{sing}(X)$ of an A_∞-space X is an A_∞-algebra.

More trivially, we can include:

$$\text{Assoc. Alg} \subset \text{DGA} \subset A_\infty \text{ Alg}$$

in which case $m_n = 0$ for all $n \geq 3$.

Definition. A *morphism* of A_∞-algebras $f : A \to B$ is a collection of graded vector space maps

$$f_n : A^{\otimes n} \to B$$

of degree $n-1$, such that

1. $f_1 m_1 = m_1 f_1$ that is, f_1 is a morphism of complexes.

2. for $n \geq 1$

$$\sum_{p+q+r} (-1)^{p+qr} f_k(id_A^{\otimes p} \otimes m_q^A \otimes id_A^{\otimes r}) - \sum_{\substack{j \geq 2 \\ i_1+\cdots+i_j=n}} (-1)^{\epsilon} m_k^B(f_{i_1}, \cdots, f_{i_k}) = \partial(f_n)$$

We call f a quasi-isomorphism if f_1 is a quasi-isomorphism, and we call f strict if $f_i = 0$ for every $i \neq 1$.

The composite of two morphisms is given by

$$(f \circ g)_n = \sum (-1)^{\epsilon} f_r \circ (g_{i_1} \otimes \cdots \otimes g_{i_r})$$

The Homotopy Transfer Theorem

Theorem (Kadeishvili, 1980)**.** *Let*

$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (V, d_V)$$

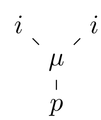
be a retract, ie

$$\begin{aligned} id_A - ip &= d_a h + h d_a \\ pi = id_V & \end{aligned}$$

If (A, d_A) is a DGA, then (V, d_V) inherits an A_∞-algebra structure such that i and p are A_∞ quasi-isomorphisms.

The basic idea of the proof is as follows. We have a natural idea for what the 'multiplication' m_2 should be:

$$m_2 = p \circ \mu \circ (i, i) =$$



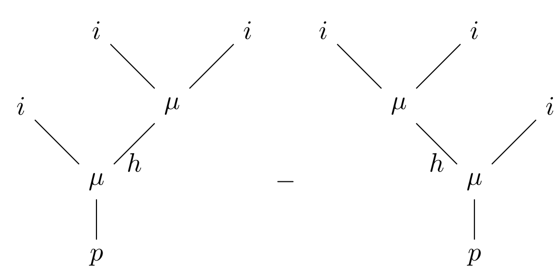
This isn't associative, but forming the associator:

$$p\mu(ip\mu(i,i), i) - p\mu(i, ip\mu(i,i)$$

we see that the obstruction to associativity is, in some sense, the difference between ip and id_A. We have a homotopy between these two elements, and we try using this in place of ip.

$$m_3 = p\mu(h\mu(i,i), i) - p\mu(i, h\mu(i,i))$$

Or, graphically



We can compute the boundary, yielding:

$$\partial(m_3) = -m_2(m_2, id) + m_2 \circ (id, m_2)$$

Iterating this construction[15] yields the desired structure.

There is a stronger version of Kadeishvili's theorem[16]:

Theorem (Kadeishvili)**.** *Let*

$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (V, d_V)$$

be a homotopy retract, ie

$$\begin{aligned} id_A - ip &= d_a h + h d_a \\ pi = id_V & \end{aligned}$$

If (A, d_A) is an A_∞-algebra, then (V, d_V) inherits an A_∞-algebra structure $\{m_n\}_{n\geq 2}$ such that i extends to an A_∞ quasi-isomorphism.

Definition. We call an A_∞-algebra with $m_1 = 0$ *minimal.* We call $H^*(A_\bullet)$ with the A_∞-structure inherited from the theorem a *minimal model* for A.

There are topological versions of this theorem as well, but even in this algebraic form, it can be given topological meaning.

Definition. Let $(A_\bullet, d)$ be a DGA. $x, y, z \in H^*(A_\bullet, d)$ such that

$$x \cup y = 0 = y \cup z$$

This implies that there exist $a, b \in A_\bullet$ such that

$$\begin{aligned} da &= x \cup y(-1)^{|x|} \\ db &= y \cup z(-1)^{|y|} \end{aligned}$$

So we define the *Triple Massey Product* [17]

$$\langle x, y, z\rangle_3 = (-1)^{|x|} x \cup b + (-1)^{|a|} a \cup z$$

Lemma. *Up to a sign,*

$$\langle x, y, z\rangle_3 = m_3(x, y, z)$$

where defined.[18]

Proof. We write $A_n = B_n \oplus H_n(A_\bullet) \oplus B_{n-1}$ (coefficients assumed to be in a field). Then we can write a homotopy explicitly treating i as the inclusion on to the factor H_n.

$$\begin{aligned} h : A_n &\to A_n + 1 \\ h &= \begin{cases} 0 & \text{on } H_n \oplus B_{n-1} \\ id & \text{on } B_n \end{cases} \end{aligned}$$

[15] More precisely, to each tree given by concatenating multiplications, we assign a copy of μ for every vertex, h for every interal edge, i for every incoming half-edge, and p for the outgoing half-edge. For a tree T call this map m_T. Then m_n is given by:

$$m_n = \sum_{T \in \text{vertices}(K_n)} \pm m_T$$

[16] Actually, there is one even stronger than this, which asserts the same result in the case of a general homotopy equivalence.

[17] More generally, one can define n-ary operations on cohomology elements satisfying some (slightly more esoteric) conditions $\langle -, \cdots, -\rangle_n$.

[18] And, moreover, we actually have

$$\langle -, \cdots, -\rangle_n = \pm m_n$$

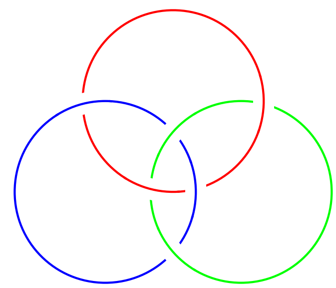
Given x, y, z as in the definition, we can choose a and b to have the additional property such that $hd(a) = a$ and $hd(b)$. Then

$$\begin{aligned} m_3(x,y,z) &= -p\mu(i(x), h\mu(i(y), i(z))) + p\mu(h\mu(i(x), i(y)), i(z)) \\ &= -x \cup h(y \cup z) + h(x \cup y) \cup z \\ &= x \cup (-1)^{|y|} x \cup h(db) + (-1)^{|x|} h(da) \cup z \\ &= x \cup (-1)^{|y|} x \cup b + (-1)^{|x|} b \cup z \\ &= (-1)^{|x|+|y|} \langle x, y, z \rangle_3 \end{aligned}$$

□

The massey products also admit some geometric intuition which is lacking in the construction of the m_i[19].

[19] As the conventional example, we have the Borromean Rings, three linked circles embedded into 3-space:



If we embed the borromean rings into S^3, and take the complement $S^3 \setminus B$, we get three generators of the first cohomology group corresponding to the three rings. However, since their pairwise linking numbers are zero, their cup products are as well. However, the third massey product of these three generators is non-zero, and in some sense represents a "three-fold linking number".

A_∞ Categories

An A_∞-category can be thought of as generalization of the structure we observed in A_∞-algebras to a multi-object system, to wit:

Definition. A *non-unital A_∞ Category* $\mathcal{A}$ consists of the following data:

- A set of objects $\mathrm{Ob}(\mathcal{A})$
- A graded vector space $\hom_{\mathcal{A}}(X_0, X_1)$ for every pair of objects X_0 X_1.
- Composition maps for all $d \geq 1$

  $$\mu^d_{\mathcal{A}} : \hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes \hom_{\mathcal{A}}(X_0, X_1) \to \hom_{\mathcal{A}}(X_0, X_d)$$

  of degree $2 - d$ such that

  $$\sum_{m,n} (-1)^{\square_n} \mu^{d-m+1}_{\mathcal{A}} (a_d, \ldots, a_{n+m+1}, \mu^m_{\mathcal{A}}(a_{n+m}, \ldots, a_{n+1}), a_n, \ldots, a_0) = 0$$

  where the sum is taken over $1 \leq m \leq d$ and $0 \leq n \leq d - m$ and the symbol $\square_n = |a_1| + \cdots + |a_n| - n$.

Definition. A non-unital A_∞-functor between 2 non-unital A_∞-categories $\mathcal{A}$ and $\mathcal{B}$ is a map

$$\mathcal{F} : \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{B})$$

and multilinear maps of degree $1 - d$ for every $d \geq 1$

$$\mathcal{F}^d : \hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes \hom_{\mathcal{A}}(X_0, X_1) \to \hom_{\mathcal{B}}(\mathcal{F}X_0, \mathcal{F}X_d)$$

such that

$$\begin{aligned} &\sum_{r \geq 1} \sum_{s_1 + \cdots + s_r = d} \mu^r_{\mathcal{B}}(\mathcal{F}^{s_r}(a_d, \ldots, a_{d-s_r+1}, \cdots, \mathcal{F}^{s_1}(a_{s_1}, \ldots, a_1)) \\ = &\sum_{m,n} (-1)^{\square_n} \mathcal{F}^{d-m+1} (a_d, \ldots, a_{n+m+1}, \mu^m_{\mathcal{A}}(a_{n+m}, \ldots, a_{n+1}), a_n, \ldots, a_0) \end{aligned}$$

The composition of functors is given by

$$
\begin{aligned}
& (\mathcal{G} \circ \mathcal{F})^d(a_d, \cdots, a_1) \\
= & \sum_r \sum_{s_1+\cdots+s_r=d} \mathcal{G}^r\left(\mathcal{F}^{s_r}(a_d, \ldots, a_{d-s_r+1}), \ldots, \mathcal{F}^{s_1}(a_{s_1}, \ldots, a_1)\right)
\end{aligned}
$$

Remark. The collection of non-unital functors $\mathcal{A} \to \mathcal{B}$, forms a non-unital A_∞ category nu-fun$(\mathcal{A}, \mathcal{B})$.

Triangulated A_∞ Categories

Gustavo Jasso

Motivation

Fix a field k, and let $\mathcal{A}$ be a k-category with one object[20]. We have the Yoneda embedding

[20] That is, a k-algebra

$$\mathcal{A} \hookrightarrow \operatorname{mod} \mathcal{A}$$

into the k-category of (right) $\mathcal{A}$ modules. This is abelian, and in some sense a 'nice' category to embed into. However, we would like the functor $\operatorname{Ext}^\ell_{\mathcal{A}}(-, M)$ to be representable, and it is not.

We can fix this issue by taking a further embedding

$$\operatorname{mod} \mathcal{A} \hookrightarrow D(\mathcal{A})$$

into the derived category of $\mathcal{A}$. In this setting we then have that

$$\operatorname{Ext}^\ell_{\mathcal{A}}(-, M) \cong \operatorname{Hom}_{D(\mathcal{A})}(-, M[\ell])|_{\operatorname{mod} \mathcal{A}}$$

Problem. $\operatorname{mod} \mathcal{A}$ is a k-category with the **property** of being abelian, whereas $D(\mathcal{A})$ is a k-category with the **structure** of a triangulation.

Fundamental observation: $D(\mathcal{A}) \cong H^0$(a triangulated A_∞ cat) and $D_{\mathrm{perf}}(\mathcal{A}) \cong H^0(Tw\mathcal{A})$ up to idempotent completion, where $Tw\mathcal{A}$ is the triangulated A_∞ category of twisted $\mathcal{A}$-modules.

A_∞ Categories and A_∞ Functors

Recall. From the last talk: A non-unital A_∞ category $\mathcal{A}$ consists of the following data:

- A set of objects $\operatorname{Ob}(\mathcal{A})$
- A graded vector space $\hom_{\mathcal{A}}(X_0, X_1)$ for every pair of objects X_0 X_1.
- Composition maps for all $d \geq 1$

$$\mu^d_{\mathcal{A}} : \hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes \hom_{\mathcal{A}}(X_0, X_1) \to \hom_{\mathcal{A}}(X_0, X_d)$$

of degree $2-d$ such that

$$\sum_{m,n}(-1)^{\square_n}\mu_{\mathcal{A}}^{d-m+1}\left(a_d,\ldots,a_{n+m+1},\mu_{\mathcal{A}}^{m}(a_{n+m},\ldots,a_{n+1}),a_n,\ldots,a_0\right)=0$$

where the sum is taken over $1\leq m\leq d$ and $0\leq n\leq d-m$ and the symbol $\square_n=|a_1|+\cdots+|a_n|-n$.

We can use this to provide other interesting constructions:

Definition. The *opposite category of* $\mathcal{A}$, $\mathcal{A}^{opp}$, is given by

- $\mathrm{Ob}(\mathcal{A}^{opp})=\mathrm{Ob}(\mathcal{A})$
- $\hom_{\mathcal{A}^{opp}}(X_0,X_1)=\hom_{\mathcal{A}}(X_1,X_0)$
- $\mu^d_{\mathcal{A}^{opp}}(a_d,\ldots,a_1)=(-1)^{\square_d}\mu^d_{\mathcal{A}}(a_1,\ldots,a_d)$

Definition. The *cohomological catgeory of* $\mathcal{A}$, $H(\mathcal{A})$, is given by

- $\mathrm{Ob}(H(\mathcal{A}))\cong\mathrm{Ob}(\mathcal{A})$
- $H(\mathcal{A})(X_0,X_1)=H(\hom_{\mathcal{A}}(X_0,X_1),\mu^1_{\mathcal{A}})$
- $[a_2]\cdot[a_1]=(-1)^{|a_1|}[\mu^2_{\mathcal{A}}(a_2,a_1)]$

$H(\mathcal{A})$ then becomes a graded **non-unital** k-category[21].

[21] In a way similar to the Homotopy Transfer Theorem from the last talk, we could endow $H(\mathcal{A})$ with the structure of an A_∞ category once again, with possibly non-trivial higher compositions. For our purposes here, however, it is enough to think of as a regular category.

There are several notions of unitality attached to A_∞-categories of which we will make use.

Definition. An A_∞ category $\mathcal{A}$ is *strictly unital* if, for all $X\in\mathrm{Ob}(\mathcal{A})$, there exists $e_X\in\hom^0_{\mathcal{A}}(X,X)$ such that

i. $\mu^1_{\mathcal{A}}(e_X)=0$

ii. For every $A\in\hom_{\mathcal{A}}(X_0,X_1)$,

$$(-1)^{|a|}\mu^2_{\mathcal{A}}(e_X,a)=a=\mu^2_{\mathcal{A}}(a,e_X)$$

iii. For every $d>2$, and for all $0\leq n<d$

$$\mu^d_{\mathcal{A}}(a_{d-1},ldots,a_{n+1},e_{X_n},a_n,\ldots,a_1)=0$$

Remark. Fukaya categories are, in general, **not** strictly unital. They do, however satisfy the next notion of unitality, which is rather weaker.

Definition. An A_∞ category $\mathcal{A}$ is called *cohomologically unital* (or *c-unital*) if $H(\mathcal{A})$ is a unital graded k-category.

Recall. A non-unital A_∞-functor between 2 non-unital A_∞-categories $\mathcal{A}$ and $\mathcal{B}$ is a map

$$\mathcal{F} : \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{B})$$

and multilinear maps of degree $1-d$ for every $d \geq 1$

$$\mathcal{F}^d : \hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes \hom_{\mathcal{A}}(X_0, X_1) \to \hom_{\mathcal{B}}(\mathcal{F}X_0, \mathcal{F}X_d)$$

such that

$$\begin{aligned} & \sum_{r\geq 1} \sum_{s_1+\cdots+s_r=d} \mu^r_{\mathcal{B}}(\mathcal{F}^{s_r}(a_d, \dots, a_{d-s_r+1}, \cdots, \mathcal{F}^{s_1}(a_{s_1}, \dots, a_1)) \\ = \; & \sum_{m,n} (-1)^{\square_n} \mathcal{F}^{d-m+1}(a_d, \dots, a_{n+m+1}, \mu^m_{\mathcal{A}}(a_{n+m}, \dots, a_{n+1}), a_n, \dots, a_0) \end{aligned}$$

Definition. For $\mathcal{A}$ and $\mathcal{B}$ strictly unital A_∞ categories, a functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is called *strictly unital* if

- For every $X \in \mathcal{A}$ $\mathcal{F}^1(e_X) = e_{\mathcal{F}(X)}$
- For any $d \geq 2$

$$\mathcal{F}^d(a_{d-1}, \dots, a_{n+1}, e_{X_n}, a_n, \dots, a_1) = 0$$

Definition. For $\mathcal{A}$ and $\mathcal{B}$ c-unital A_∞ categories, a functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is *c-unital* if the induced functor $H(\mathcal{F})$ is unital[22].

[22] Notice that for both categories and functors, it is immediate from the definitions that being strictly unital implies being c-unital.

Definition. Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ be a functor of A_∞ categories:

i. $\mathcal{F}$ is *cohomologically full and faithful* if $H(\mathcal{F})$ is full and faithful.

ii. $\mathcal{F}$ is a *quasi-isomorphism* if $H(\mathcal{F})$ is an isomorphism.

iii. $\mathcal{F}$ is a *quasi-equivalence* if $H(\mathcal{F})$ is an equivalence of categories.

Proposition. *The following statements hold*

i. If $\mathcal{A}$ and $\mathcal{B}$ are strictly unital, the category nu-fun$(\mathcal{A}, \mathcal{B})$ is strictly unital.

ii. If $\mathcal{A}$ and $\mathcal{B}$ are c-unital, the category nu-fun$(\mathcal{A}, \mathcal{B})$ is c-unital.

Additionally, we can define:

Definition. There is a subcategory fun$(\mathcal{A}, \mathcal{B}) \subset$ nu-fun$(\mathcal{A}, \mathcal{B})$, which is the c-unital A_∞ category of c-unital A_∞ functors.

A_∞ Modules

Our objective is to generalize the Yoneda embedding mention in the first section of this talk.

Recall. Ch(k) is the *dg category of cochain complexes* of vector spaces.

- Ob(Ch(k)) are cochain complexes
- For each $i \in \mathbb{Z}$, $\mathrm{Hom}(X,Y)^i$ is given by degree i morphisms $X \to Y$.
- For each $i \in \mathbb{Z}$ the hom complex $\mathrm{Hom}(X,Y) = (\mathrm{Hom}(X,Y)^\bullet, d)$ has differential given by

$$\begin{array}{rcl} \mathrm{Hom}(X,Y) & \stackrel{d}{\to} & \mathrm{Hom}(X,Y)^{i+1} \\ f & \mapsto & f^{i+1} \circ d_X + (-1)^i d_Y \circ f^i \end{array}$$

Remark. Ch(k) can be viewed as a strictly unital A_∞-category.

Definition. For $\mathcal{A}$ a c-unital A_∞ category, we define two *categories of $\mathcal{A}$-modules*:

$$\begin{array}{ccc} \text{nu-mod}\mathcal{A} & = & \text{nu-fun}(\mathcal{A}^{opp}, \mathrm{Ch}(k)) \\ \cup & & \\ \mathrm{mod}\,\mathcal{A} & = & \mathrm{fun}(\mathcal{A}^{opp}, \mathrm{Ch}(k)) \end{array}$$

We then have the *Yoneda Embedding*

$$\begin{array}{ccc} \mathcal{A} & \stackrel{\text{Yoneda}}{\hookrightarrow} & \mathrm{mod}\,\mathcal{A} \\ X & \mapsto & \hom_{\mathcal{A}}(-,X) \end{array}$$

Remark. The Yoneda embedding is c-unital, and cohomologically full and faithful.

Triangulated A_∞ Categories

Let $\mathcal{A}$ be a c-unital A_∞ category.

Definition. Let $\mu \in \mathrm{mod}\,\mathcal{A}$. A *quasi-representative* of μ is a pair $(Y, [t])$, where $Y \in \mathcal{A}$ and

$$[t] : \hom_{\mathcal{A}}(-,Y) \stackrel{\sim}{\to} \mu$$

in $H^0(\mathrm{mod}\,\mathcal{A})$

Definition. For $X \in \mathrm{Ob}(\mathcal{A})$, a *shift of X* is a quasi-representative $(X[1], [t])$ of the A_infty module $\hom_{\mathcal{A}}(-,X)[1]$, ie

$$[t] : \hom_{\mathcal{A}}(-,X[1]) \stackrel{\sim}{\to} \hom_{\mathcal{A}}(-,X)[1]$$

To motivate the A_∞ version of the cone construction:

Recall. If $c : X \to Y$ is a morphism of complexes, we have

$$\begin{tikzcd}
X \arrow[d, "c"] & X^{-1} \arrow[r] \arrow[d, "c"] & X^0 \arrow[r] \arrow[d, "c"] & X^1 \arrow[r] \arrow[d, "c"] & \cdots \\
Y \arrow[d] & Y^{-1} \arrow[r] \arrow[d, "\begin{pmatrix}0\\1\end{pmatrix}"] & Y^0 \arrow[r] \arrow[d, "\begin{pmatrix}0\\1\end{pmatrix}"] & Y^1 \arrow[r] \arrow[d, "\begin{pmatrix}0\\1\end{pmatrix}"] & \cdots \\
\mathrm{cone}(c) \arrow[d] & X^0 \oplus Y^{-1} \arrow[r] \arrow[d, "\begin{pmatrix}1 & 0\end{pmatrix}"] & X^1 \oplus Y^0 \arrow[r] \arrow[d, "\begin{pmatrix}1 & 0\end{pmatrix}"] & X^2 \oplus Y^1 \arrow[r] \arrow[d, "\begin{pmatrix}1 & 0\end{pmatrix}"] & \cdots \\
X[1] & X^0 \arrow[r] & X^1 \arrow[r] & X^2 \arrow[r] & \cdots
\end{tikzcd}$$

We define cone(c) to be the chain complex $\mathcal{C} = X[1] \oplus Y$ with differential

$$d_{\mathcal{C}} = \begin{pmatrix} d_{X[1]} & 0 \\ c[1] & d_Y \end{pmatrix}$$

And this sequence gives the completion of c to an exact triangle.

Definition. Let $c \in \hom^0_{\mathcal{A}}(Y_0, Y_1)$ such that $\mu^1_{\mathcal{A}}(c) = 0$[23]. The *abstract mapping cone of* c is the A_∞ module $\mathcal{C} = \mathrm{Cone}(c)$ given by

$$\mathcal{C}(X) = \hom_{\mathcal{A}}(X, Y_1)[1] \oplus \hom_{\mathcal{A}}(X, Y_0)$$

Where

$$\mu^d_{\mathcal{C}}\left((b_0, b_1), a_{d-1} \ldots, a_1\right) = \\ \left(\mu^d_{\mathcal{A}}(b_0, a_{d-1} \ldots, a_1), \mu^d_{\mathcal{A}}(b_1, a_{d-1}, \ldots, a_1) + \mu^{d+1}_{\mathcal{A}}(c, b_0, a_{d-1}, \ldots, a_1)\right)$$

gives the required structure on the image[24].

In $H(\mathrm{mod}\,\mathcal{A})$, we then get

$$\begin{tikzcd}
\hom_{\mathcal{A}}(-, Y_0) \arrow[rr, "\text{Yoneda}"] & & \hom_{\mathcal{A}}(-, Y_1) \arrow[dl, "i"] \\
& \mathrm{Cone}(c) \arrow[ul, "\pi"] \arrow[u, phantom, "(*)"] &
\end{tikzcd}$$

where

$$\begin{aligned} i &\in \hom^0_{\mathrm{mod}\,\mathcal{A}}(\hom_{\mathcal{A}}(-, Y_1), \mathcal{C}) \\ \pi &\in \hom^1_{\mathrm{mod}\,\mathcal{A}}(\mathcal{C}, \hom_{\mathcal{A}}(-, Y_0)) \end{aligned}$$

[23] That is, such that c defines a morphism in $H(\mathcal{A})$.

[24] Notice that in the lowest degree case, we retrieve almost the classical cone construction.

$$\mu^1_{\mathcal{C}}((b_0, b_1) = \left(\mu^1_{\mathcal{A}}(b_0), \mu^1_{\mathcal{A}}(b_1) + \mu^2_{\mathcal{A}}(c, b_0)\right)$$

Indeed, taking A to be a dg category, for fixed X we get complexes $C = \hom_{\mathcal{A}}(X, Y_1)$ and $D = \hom_{\mathcal{A}}(X, Y_0)$, and c gives a morphism between these complexes, in this case, $\mu^1_{\mathcal{C}}$ can be viewed as a differential on the cone $C[1] \oplus D$, and is precisely equal to

$$\mu^1_{\mathcal{C}}(b_0, b_1) = (d_C(b_0), d_D(b_1) + c(b_0))$$

and

$$i^1(b_1) \quad = \quad (0, (-1)^{|b_1|} b_1$$
$$\pi^a(b_0, b_1) = (-1)^{|b_0|} b_0$$

Definition. An *exact triangle* in $\mathcal{A}$ is a diagram in $H(\mathcal{A})$

$$\begin{tikzcd} Y_0 \arrow[rr, "{[c_1]}"] & & Y_1 \arrow[dl] \\ & Y_2 \arrow[ul] & \end{tikzcd}$$

which becomes isomorphic to a diagram $(*)$ under the Yoneda embedding.

Definition. $\mathcal{A}$ is *triangulated* if

- $\mathrm{Ob}(\mathcal{A}) \neq \emptyset$
- Every $[c] \in H^0(\mathcal{A})$ can be completed to an exact triangle[25].
- For every $Y \in \mathrm{Ob}(\mathcal{A})$ there exists a $\tilde{Y} \in \mathrm{Ob}(\mathcal{A})$ such that $\tilde{Y}[1] \cong Y$ in $H^0(\mathcal{A})$.

[25] If all cones exist, can notice that

$$\mathrm{Cone}(Y \xrightarrow{e_Y} Y)$$

is a zero object in $H^0(\mathcal{A})$, and that, then,

$$\mathrm{Cone}(Y \to \mathrm{Cone}(Y \xrightarrow{e_Y} Y))$$

is a shift of Y

Theorem. *If $\mathcal{A}$ is a triangulated A_∞ category, then $H^0(\mathcal{A})$ is a triangulated k-category.*

Example.
- $\mathrm{Ch}(k)$ is a triangulated A_∞ category.
- $\mathrm{mod}\,\mathcal{A}$ is a triangulated A_∞ category.
- more generally, $\mathrm{fun}(\mathcal{A}, \mathcal{B})$ where $\mathcal{B}$ is a triangulated A_∞ category is itself a triangulated A_∞ category.

Definition. Let $\mathcal{B}$ be a triangulated A_∞ category, and $\mathcal{A}$ be a full A_∞ subcategory. We denote by $\langle \mathcal{A} \rangle_{\mathcal{B}}$ the smallest strictly full triangulated subcategory of $\mathcal{B}$ containing $\mathcal{A}$, and we call it *the triangulated subcategory of $\mathcal{B}$ generated by $\mathcal{A}$*.

Definition. $\mathcal{A} \neq \emptyset$. A *triangulated envelope* of $\mathcal{A}$ is a cohomologically fully faithful functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ with $\mathcal{B}$ a triangulated A_∞ category such that $\langle \mathcal{F}(\mathcal{A}) \rangle_{\mathcal{B}} = \mathcal{B}$[26].

[26] Triangulated envelopes exist. We can take the Yoneda embedding

$$Y : \mathcal{A} \hookrightarrow \mathrm{mod}\,\mathcal{A}$$

and then take a triangulated envelope of $\mathcal{A}$ to be $\langle Y(\mathcal{A}) \rangle_{\mathrm{mod}\,\mathcal{A}}$.

Part II

Fukaya Categories

Introduction to Symplectic Manifolds

Max Körfer

The primary goal of this talk will be to introduce basic notions related to symplectic manifolds, and to provide motivation (from Kähler Geometry and from Physics) as to why symplectic geometry is a natural topic of study.

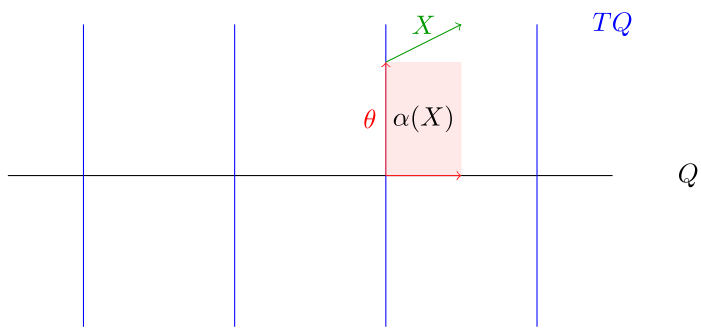
Symplectic Manifolds

Definition. A *symplectic manifold* (M, ω) is a smooth manifold M with a closed, non-degenerate[27] 2-form $\omega \in \Omega^2(M)$ called the *symplectic form* of M.

Example (Cotangent Bundles)**.** Let Q be a manifold, and let $p : T^*Q \to Q$ be the projection. We can define[28] the canonical 1-form $\alpha \in \Omega^1(T^*Q)$ by the condition that, for a point $(x, \theta) \in T^*Q$ and for $X \in T_xQ$, we have

$$\alpha(X) = \langle \theta, p_* X \rangle$$

Pictorially:



We could equivalently define α by the property that, for any $\theta \in \Omega^1(Q)$, viewed as a map $\theta : Q \to T^*Q$, we have $\theta^*\alpha = \theta$.

We take $\omega = d\alpha$. In local coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ we have that

$$\alpha = \sum_i p_i dq_i \qquad \omega = \sum_i dp_i \wedge dq_i$$

So we can immediately verify that (T^*Q, ω) is a symplectic manifold.

[27] By non-degenerate, we mean that the map of bundles

$$\tilde{\omega} : TM \to T^*M$$

induced by ω is an isomorphism. Alternately, we could say that the bilinear form induce by ω on T_xM is non-degenerate for all x.

[28] There are several equivalent conditions that define the canonical 1-form. All three conditions listed here are sufficient for that purpose.

As it turns out, the symplectic manifold T^*Q from the example is, in some sense, the prototypical example of a symplectic manifold:

Theorem (Darboux)**.** *Let (M, ω) be a symplectic manifold. Then locally, around any point $x \in M$, there are local coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ such that*

$$\omega = \sum_i dp_i \wedge dq_i$$

Proposition (Basic Observations)**.** *For (M, ω) a symplectic manifold:*

1. $\dim(\omega) = 2n$
2. $\frac{1}{n!}\omega^n$ *is an oriented volume form*
3. $[\omega^k] \in H^{2k}(M)$ *are non-trivial*

Proof. (1) and (2) follow from linear algebra and an application of the determinant, respectively[29]. For (3), we note that

$$[\omega^n] \cap [M] = \int_M \omega^n \neq 0$$

so that $[\omega] \neq 0$, and (3) follows. $\square$

[29] Since have stated Darboux's theorem already, one might be tempted to say that (1) follows from the theorem. However, (1) is a far more elementary/primary observation than Darboux's theorem.

Definition. Let (M, ω) be a symplectic manifold. An immersion

$$i : L \hookrightarrow M$$

is called *Lagrangian* (or a *lagrangian submanifold*) if, for any $x \in L$

$$i_* T_x L = (i_* T_x L)^{\perp}$$

with respect to the inner product induced by ω.

Remark. We can immediately see that $\dim L = \dim M - \dim L$, so that $\dim L = \frac{1}{2} \dim M$. To show that an immersion is Lagrangian, it suffices to check that L is half-dimensional, and ω pulls back to 0 on L.

Example. Let Q be a manifold, $\theta \in \Omega^1(Q)$, viewed as a map $\theta : Q \to T^*Q$. Let $p : T^*Q \to Q$. Then we have the inclusion

$$i : \mathrm{Graph}(\theta) \hookrightarrow T^*Q$$

and can compute that

$$\begin{aligned} i^*\omega &= (\theta p)^*\omega \\ &= p^*\theta^*\omega \\ &= p^*\theta^* d\alpha \\ &= p^* d\theta \end{aligned}$$

So we can conclude $\mathrm{Graph}(\theta)$ is lagrangian if and only if $d\theta = 0$.

Example. Let (M, ω_M) and (N, ω_N) be symplectic manifolds, and $f : M \to N$ smooth. We then have the inclusion

$$i : \mathrm{Graph}(f) \hookrightarrow M \times (-N)$$

with the symplectic form on the target defined by

$$\omega_{M\times(-N)} = p_M^* \omega_M - p_N^* \omega_N$$

We can then compute that

$$\begin{aligned} i^*\omega_{M\times(-N)} &= i^*(p_M^*\omega_M - p_N^*\omega_N) \\ &= p_M^*\omega_M - p_N^* f^*\omega_N \\ &= p_M^*(\omega_M - f_N^*) \end{aligned}$$

We can then conclude that $\mathrm{Graph}(f)$ is lagrangian if and only if $\dim M = \dim N$ **and** $f^*\omega_N = \omega_M$.

Theorem (Weinstein, Local Neighborhood Theorem)**.** *Let (M, ω) be a symplectic manifold. $L \subset M$ lagrangian, L compact, and $\partial L = \emptyset$. Then there exist neighborhoods $U \subset M$, $V \subset T^*L$ of L[30] and a symplectomorphism[31] $\phi : U \to V$ making*

$$\begin{tikzcd} & U \arrow[dr] & \\ L \arrow[ur, hook] \arrow[rr, hook] & & V \end{tikzcd}$$

commute.

[30] For the definition of V, we mean a neighborhood of the zero section $L \subset T^*L$

[31] We say that a morphism of symplectic manifold $f : M \to N$ is a symplectomorphism when $f^*\omega_N = \omega_M$.

Definition. Let (M, ω) be a symplectic manifold.

1. For a *hamiltonian function* $H : M \to \mathbb{R}$ define the *hamiltonian vector field* $X_H \in \mathcal{X}(M)$ by the condition that

$$\iota_{X_H}\omega = -dH$$

2. For $f, g \in C^\infty(M)$ define the *poisson bracket*

$$\{f, g\} = \omega(X_f, X_g) = X_g f$$

Remark. We can compute the lie derivative of ω along a hamiltonian vector field:

$$\mathcal{L}_{X_H}\omega = d\iota_{X_H}\omega = -d^2H = 0$$

Showing that ω is invariant under hamiltonian flows[32]

[32] A hamiltonian flow for H is simply a flow through the vector field X_H. More precisely, a hamiltonian flow is a function

$$\Phi : M \times \mathbb{R} \to M$$

such that

$$\frac{d}{dt}|_{t=0}\Phi(x,t) = (X_H)_x$$

Proposition. *For (M, ω) a symplectic manifold,*

1. $C^\infty(M)$ is a poisson algebra with $\{-,-\}$

2. *$C^\infty(M) \to \mathcal{X}(M)$ given by $H \mapsto X_H$ is an anti-homomorphism of Lie Algebras.*

Example (Geodesic Flow)**.** Let Q be a Riemannian Manifold with metric g. We have an isomorphism induced by g

$$\tilde{g} : TQ \to T^*Q$$

so we can view TQ as symplectic with form $\tilde{g}^*\omega$. If we take the hamiltonian function

$$H(X) = \frac{1}{2}g(X,X) =: \frac{1}{2}X^2$$

The hamiltonian flow generated by H is the geodesic flow, so that the flow line through (x, X) is given by (γ_t, γ_t') where γ is the geodesic through x with initial velocity X.

Example. Let Q be a manifold,with a function $S : Q \to \mathbb{R}$. Then we have a hamiltonian function $p^*S : T^*Q \to \mathbb{R}$. We can write its hamiltonan vector field X_S in local coordinates as

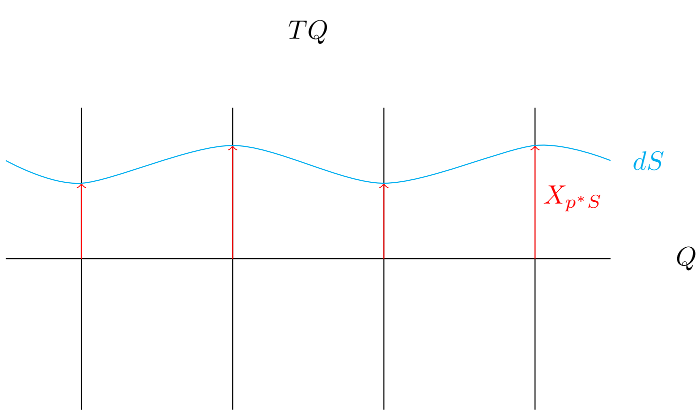
$$-\sum_i \frac{\partial p^*S}{\partial q^i} \frac{\partial}{\partial p_i}$$

and have

$$dS = \sum_i \frac{\partial S}{\partial q^i} dq^i$$

Taking the Hamiltonian flow Φ_t through X_S, we see that Φ_1[33] sends the zero section to dS. Pictorially:

[33] Note that this is a symplectomorphism, and thus sends lagrangian submanifolds to lagrangian submanifolds.



Symplectic Geometry and Kähler Geometry

Definition. A *Kähler manifold* (M, h) is a complex manifold M equipped with a hermitian metric[34] h such that $d\omega = 0$ for $\omega = \Im(h)$.

[34] That is, a form $h \in \Gamma(T^*X \otimes \overline{T^*M})$ such that $h(a, \overline{a}) \geq 0$ and $h(a, \overline{b}) = \overline{h(b, \overline{a})}$ for all $a, b \in T^*M$.

Remark. We have that

$$\omega(X,Y) = \Im(h(X,Y)) = \Im(\overline{h(Y,X)}) = -\Im(h(Y,X)) = -\omega(Y,X)$$

And, letting $\mathcal{J}$ be the almost complex structure given by multiplication by i, we have

$$\omega(\mathcal{J}X, X) = \Im(h(\mathcal{J}X, X)) = \Im(ih(X, X)) = \Re(h(X, X)) > 0$$

For $X \neq 0$.

Proposition (Characterizations of Kähler Manifolds)**.** *The following are equivalent:*

1. (M, h) *is Kähler*

2. $d\omega = 0$

3. $\nabla^{LC}\mathcal{J} = 0$ *where* ∇^{LC} *is the Levi-Civita Connection.*

4. $\nabla^{CH} = \nabla^{LC}$ *where* ∇^{CH} *is the Chern connection.*

Example (Smooth Complex Projective Varieties)**.** Take $S^{2n+1} \subset \mathbb{C}^{n+1}$ with the standard metric. The unitary group $U(1)$ acts on S^{2n+1} isometrically, so we can form the quotient $\mathbb{C}P^n = \S^{2n+1}/U(1)$, and it inherits a metric called the *Fubini-Study metric*. It can be show that $\mathbb{C}P^n$ equipped with this metric is a Kähler Manifold.

Moreover, suppose (M, h) is a Kähler manifold, and $i : N \hookrightarrow M$ is a submanifold. Then $i^*\omega_M = \omega_N$, so that $d\omega_N = 0$, making N into a Kähler Manifold.

Remark. We can define a Riemannian metric on any Kähler Manifold by

$$g(X, Y) = \Re(h(X, Y))$$

We also see that

$$g(\mathcal{J}X, Y) = \Re(h(\mathcal{J}X, Y) = \Re(ih(X, Y) - h(X, Y))$$

So that we can retrieve the imaginary part of h from g and $\mathcal{J}$, and thus, we can retrieve h from g and $\mathcal{J}$.

Definition. Let M be a manifold

1. An *almost complex structure* on M is a complex structure on TM. That is, $\mathcal{J} : TM \to TM$ such that $\mathcal{J}^2 = -id$.

2. $\mathcal{J}$ is called *integrable* if $(M, \mathcal{J})$ is locally isomorphic to C^n.

3. An almost complex structure is called *compatible* if

$$\omega(X, Y) = g(X, \mathcal{J}Y)$$

   for any $X, Y \in TM$.

Proposition. *Let (M, ω) be a symplectic manifold. The space of almost complex structures compatible with ω is non-empty and contractible.*

Sketch of proof. Equipping M with a metric $\tilde{g}$ allows us to define an automorphism $\tilde{\mathcal{J}}$ by $\omega(X, Y) = \tilde{g}(X, \tilde{\mathcal{J}}Y)$. As a result, we have immediately that $\mathcal{J}^* = -\mathcal{J}$.

If we consider $\tilde{\mathcal{J}} = A\mathcal{J}$ where $A^* = A$ and $\mathcal{J}$ is orthogonal (WRT $\tilde{g}$), we can show that $\mathcal{J}^2 = -1$. If we set $g = \tilde{g}(-, A-)$ we see that $(\omega, g, \mathcal{J})$ are compatible. □

Theorem (Newlander-Nirenber)**.** *Let M be a manifold, and $\mathcal{J}$ an almost complex structure. Then $\mathcal{J}$ is integrable if and only if*[35]

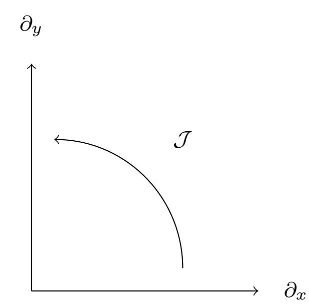
$$\left[\Gamma(T^{(1,0)}), \Gamma(T^{(1,0)})\right] \subset T^{(1,0)}$$

[35] Equivalently, we could say that the Lie bracket preserves the splitting $T = T^{(1,0)} \oplus T^{(0,1)}$ or that

$$\left[\Gamma(T^{(0,1)}), \Gamma(T^{(0,1)})\right] \subset T^{(0,1)}$$

Example (Oriented Surface)**.** Let M be an oriented surface, and g be any Riemannian metric. Set $\omega = \sqrt{\det(g)}dxdy$, and let $\mathcal{J}$ be the almost complex structure that comes from rotating the tangent space 90 degrees[36] We find that $(g, \omega, \mathcal{J})$ are compatible. Notice that, almost definitionally, any two conformally equivalent metrics give the same $\mathcal{J}$.

[36] Pictorially:



Symplectic Geometry and Physics

Example (Classical Mechanics)**.** Let Q be a configuration space[37] There is a function, called the *Lagrangian*

$$L : TQ \to \mathbb{R}$$

which characterizes the time evolution of the system, under the rule that $q : \mathbb{R} \to Q$ is a physical path if and only if

$$\int_{\mathbb{R}} \frac{d}{ds}|_{s=0} L(q_s, \dot{q}_s) dt = 0$$

for any smooth family of paths q_s equal to q at $s = 0$.

[37] That is, a manifold that, in some sense, parameterizes all the possible configurations of a physical system.

This condition is equivalent to q being a solution of the *Euler-Lagrange equations*:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i}$$

We then take the *Legendre Transform*

$$\begin{array}{rcl} TQ & \to & T^*Q \\ (q, \dot{q}) & \mapsto & \left(q, \dfrac{\partial L}{\partial \dot{q}} =: p\right) \end{array}$$

We will assume that this is an isomorphism. Applying the Legendre transform to L we get a function H such that

$$H(q,p) \quad = \quad \dot{q}p - L(q,\dot{q})$$
$$L(q,\dot{q}) = qp - H(q,p)$$

In terms of H, the Euler-Lagrange equations become

$$\begin{aligned} \dot{p} &= -\frac{\partial H}{\partial q} \\ \dot{q} &= \frac{\partial H}{\partial p} \end{aligned}$$

which are known as the *Hamilton-Jacobi* equations.

Using the usual symplectic structure on T^*Q, we can compute the hamiltonian vector field associated to H

$$X_H = -\frac{\partial H}{\partial q}\frac{\partial}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial}{\partial q}$$

so that

$$\iota_{X_H} dp \wedge dq = \frac{\partial H}{\partial q}dq - \frac{\partial H}{\partial p}dp = -dH$$

So that the condition that $q : \mathbb{R} \to Q$ be a flow line of a Hamiltonian flow for H is equivalent to the condition that f satisfy the Hamilton-Jacobi Equations[38]

Example (Lagrangian Field Theories)**.** We again begin with a manifold M, this time representing spacetime, and we assume it comes equipped with a fiber bundle $F \to M$. We define $\mathcal{F} = \Gamma(F)$ to be the *space of fields.*

The dynamics (time evolution) are now given by a Lagrangian density

$$\mathcal{L} : \mathcal{F} \to \Omega^{|\text{top}|}(M)$$

in the sense that a field $\phi \in \mathcal{F}$ is physical if and only if

$$\int_M \frac{d}{ds}|_{s=0} \left(\mathcal{L}(\phi_s)\right) = 0$$

for a smooth family of fields ϕ_s with $\phi_0 = \phi$.[39]

We then define the space

$$\Omega^{k,|\ell|}_{\text{loc}} = \lim_{\substack{\to \\ m}} \Omega^{k+1}_{\text{vert}}(J^m F) \otimes_{C^\infty} \Omega^{|\ell|}(M)$$

Where $J^k F$ is the $k^t h$ jet bundle of F[40].

We have differentials

$$\delta : \Omega^k_{\text{vert}}(J^m F) \to \Omega^{k+1}_{\text{vert}}(J^m F)$$

[38] One of the best known basic examples in physics is the Harmonic oscillator, ie, the example of a mass m attached to a spring with spring constant k. The Lagrangian for this problem is given by

$$L = \frac{1}{2}m\dot{q} - \frac{1}{2}kq^2$$

and the Hamiltonian is

$$H(q,p) = \frac{p^2}{2m} + \frac{1}{2}kq^2$$

[39] An example might be, given the trivial bundle $F = \mathbb{R} \times M \to M$,

$$\mathcal{L}(\phi) = \frac{1}{2}m|d\phi|^2 + \lambda\phi^2 d(\text{vol})$$

where vol is the volume form.

[40] For a fiber bundle $\pi : F \to M$, the $k^t h$ jet bundle is a new fiber bundle whose fibers can be thought of as germs of sections of F, quotiented by a certain equivalence relation. We consider two germs of sections equivalent in the $k^t h$ jet bundle $J^k F$ if they have the same partial derivatives up to order k.

and

$$d : \Omega^n_{\text{vert}}(J^m F) \otimes_{C^\infty} \Omega^{|\ell|}(M) \to \Omega^k_{\text{vert}}(J^{m+1}F) \otimes_{C^\infty} \Omega^{|\ell|}(M)$$

We can consider $\mathcal{L} \in \Omega^{0,|\text{top}|}_{\text{loc}}(F \times M)$, So that, under the induced differential, we have $\underline{D}\mathcal{L} \in \Omega^{1,|\text{top}|}_{\text{loc}}(F \times M)$. In this setup, the analogue of the Euler-Lagrange equations is that ϕ represents a physical field if and only if $\underline{D}\mathcal{L} = 0$.

We can write

$$\underline{D}\mathcal{L} = \delta\mathcal{L} + d\gamma$$

calling $\gamma \in \Omega^{1,|\text{top}|-1}$ the *variational 1-form.* We can then define

$$\delta\gamma =: \omega \in \Omega^{2,|\text{top}|-1}(M)$$

so that $\omega = \underline{D}(\mathcal{L} + \gamma)$ on $M \times M$.

In general, this is not a symplectic form (non-degeneracy fails whenever there are symmetries of the system). However, in some cases (eg, modding out by gauge symmetries), ω can give a symplectic form on a quotient of $M \times M$.

Example (Semiclassical Stationary States)**.** For a non-relativistic quantum particle, with position/velocity $q : \mathbb{R}^3 \to \mathbb{R}$ and $v : \mathbb{R}^3 \to \mathbb{R}$, we can define the *state space* for the system to be the hilbert space $\mathcal{H} = L^2(\mathbb{R}^3)$. The Hamiltonian operator dictating the dynamics of the system is given by

$$\hat{H} = -\frac{\hbar^2}{2m}\Delta + V$$

For some potential V. This corresponds to a hamiltonian function on $T^*\mathbb{R}^3$

$$H(p, q) = \frac{p^2}{2m} + V$$

The time evolution is governed by the *Schrödinger Equation*

$$i\hbar\frac{\partial}{\partial t}\psi = \hat{H}\psi$$

By considering separable solutions of the form

$$\psi(t) = \exp(\frac{it\hat{H}}{\hbar})\psi_0$$

we can consider the stationary equation

$$\hat{H}\psi = E\psi$$

for $E \in \mathbb{R}$

If $V = 0$, the solutions to the stationary equation are waves of the form $e^{ix\xi}$. As an approximation to solutions in the presence of a potential, we can make the WKB ansatz, namely that

$$\psi = a(x)\exp(\frac{iS(x)}{\hbar})$$

Plugging this in yields the equation:

$$\begin{aligned}(\hat{H}-E)\psi &= \big[-\frac{\hbar^2}{2m}\Delta a\psi - i\frac{\hbar}{2m}\nabla a\nabla S - i\frac{\hbar}{2m}a\Delta S \\ &\quad +\frac{1}{2m}a(\nabla S^2)\exp(\frac{iS(x)}{\hbar}) + aV)\big]\end{aligned}$$

In a first-order approximation, we get

I) $\nabla a\nabla S + \frac{1}{2}a\Delta S = 0$

I) $\frac{1}{2m}(\nabla S)^2 + V = E$

(II) is equivalent to $H(q, \nabla S) = E$, which are the Hamilton equations on the lagrangian submanifold $L = \mathrm{Graph}(dS)$.

Floer Homology I

TOBIAS DYCKERHOFF

Symplectic Geometry as Lagrange did it

The following example is taken from the paper *Memoire sur la theori des Varione des éléments de planéts et la particulier des variatins des grends axes de leur orbites*, published in 1808 by Lagrange.

We can write down the equation of motion for a planet with mass m revolving around the sun[41].

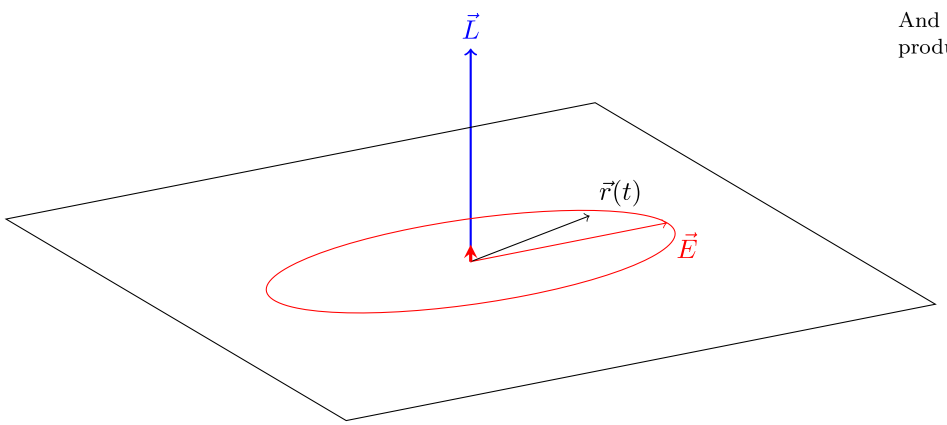
$$\frac{d^2\vec{r}}{dt^2} + \frac{1+m}{|\vec{r}|^3}\vec{r} = 0 \qquad (*)$$

where $\vec{r} = (x(t), y(t), z(t)) \in \mathbb{R}^3$. Supposing we have a solution $\vec{r}(t)$, the *angular momentum*

$$\vec{L} = \vec{r} \times (\dot{\vec{r}}m)$$

is conserved.[42]

Pictorially, our trajectories will therefore be confined to the plane perpendicular to $\vec{L}$.



The vector $\vec{E}$ in the diagram, which runs along the positive semi-major axis, is known as the *Laplace vector*.

[41] For simplicity, we normalize such that the sun is presumed to have mass 1.

[42] This can be seen by taking

$$\begin{aligned} \frac{d\vec{L}}{dt} &= \dot{\vec{r}} \times \dot{\vec{r}}m + \vec{r} \times \ddot{\vec{r}}m \\ &= \vec{r} \times \frac{d^2\vec{r}}{dt^2}m \end{aligned}$$

And noticing that, taking the cross product of $\vec{r}$ with equation (*), we get

$$\vec{r} \times \frac{d^2\vec{r}}{dt^2} = 0$$

We can, therefore, describe any solution by specifying a point in

$$\left.\begin{array}{l}\left\{(\vec{L},\vec{E}) \in \mathbb{R}^3 \times \mathbb{R}^3 \;\middle|\; \vec{L} \neq 0,\ |\vec{E}| < const,\ \langle \vec{L},\vec{E}\rangle = 0\right\} \\ \text{angle } \theta \text{ of } \vec{r}(t) \text{ at time } 0\end{array}\right\}\mathcal{E}$$

We see that $\mathcal{E}$ is a 6-dimensional manifold, and, following Lagrange, we coordinatize it by

- a –semi-major axis
- b – eccentricity
- c – epoch
- f, g, h–position of the ellipse in 3-space

This setup assumes that there is no effect from other planets on the orbit[43]. If we want to study the effect of other planets, we nned to study a perturbation of equation (*), whcih leads to a 'drift motion' of points in $\mathcal{E}$. Lagrange analyzed this problem, and arrived at the following solution.

[43] It also neglects the influence of the orbiting planet on the sun.

There exists a symplectic form on $\mathcal{E}$[44] given in coordinates by

[44] This is, in some sense, the first symplectic form ever written down.

$$\begin{aligned}\omega \quad = \quad & -\frac{na}{2} da \wedge dc - \frac{na\sqrt{1-b^2}}{2} da \wedge df - \frac{na\sqrt{1-b^2}}{2}\cos(h) da \wedge dg \\ & + \frac{na^2 b}{\sqrt{1-b^2}} db \wedge df + \frac{na^2 b}{\sqrt{1+b^2}}\cos(h) db \wedge dg - na^2\sqrt{1-b^2}\sin(h) dg \wedge dh\end{aligned}$$

where we have introduced $n := \sqrt{\frac{1+m}{a^3}}$ to ease notation.

Further, Lagrange observed that there exists a function $H \in C^\infty(\mathcal{E}, \mathbb{R})$ such that the drift motion due to the gravitational effect of other planets is given by the hamiltonian flow of the vector field X_H defined with respect to the form ω.

Floer Theory

We begin with *Floer Theory.* Let

- (M, ω) be a symplectic manifold[45]
- $H_t \in C^\infty(M \times [0,1], \mathbb{R})$ be a hamiltonian function. We get the associated hamiltonian vector field X_t and a hamiltonian diffeomorphism $\phi : X \to X$ given by integrating X_t over $t \in [0,1]$.
- L compact Lagrangian submanifold.

[45] Subject to some assumptions related to compactness/bounded geometry.

Theorem (Floer). *Assume that the symplectic area of any (immersed)*

- *2-disk with boundary in L*
- *2-sphere*

vanishes. Assume $\phi(L)$ and L intersect transversely. Then

$$\#(\phi(L) \cap L) \geq \sum_{k \geq 0} \dim H^k(L, \mathbb{F}_2)$$

Application (Arnold's Conjecture)**.** Let (N, ω) be a compact symplectic manifold. $\psi : N \to N$ a Hamiltonian diffeomorphism. Assume all fixed points of ψ are non-degenerate, then

$$\#\{\text{fixed points}\} \geq \sum_{k \geq 0} \dim H^k(N, \mathbb{F}_2)$$

Theorem $\Rightarrow$ Application. (We assume all spheres in M have trivial volume.) We can construct a new symplectic manifold

$$M = (N \times N, \pi_1^* \omega - \pi_2^* \omega)$$

And find lagrangian submanifolds[46]

$$\left.\begin{array}{l} L := N \overset{\Delta}{\subset} (N \times N, \omega) \\ \tilde{\psi}(L) = \text{graph}(\psi) \end{array}\right\}$$

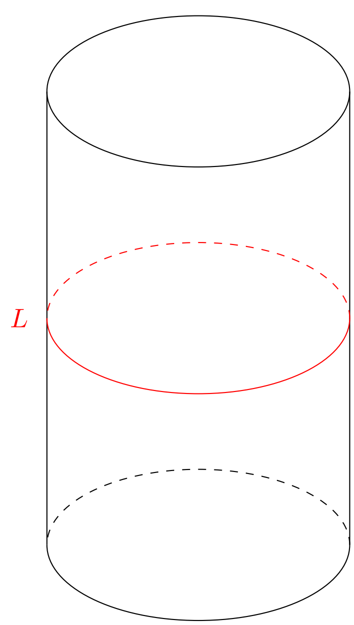
And so

$$\#\{\text{fixed points}\} = \#\{\tilde{\psi}(L) \cap L\} \geq \sum_{k \geq 0} \dim H^k(L, \mathbb{F}_2)$$

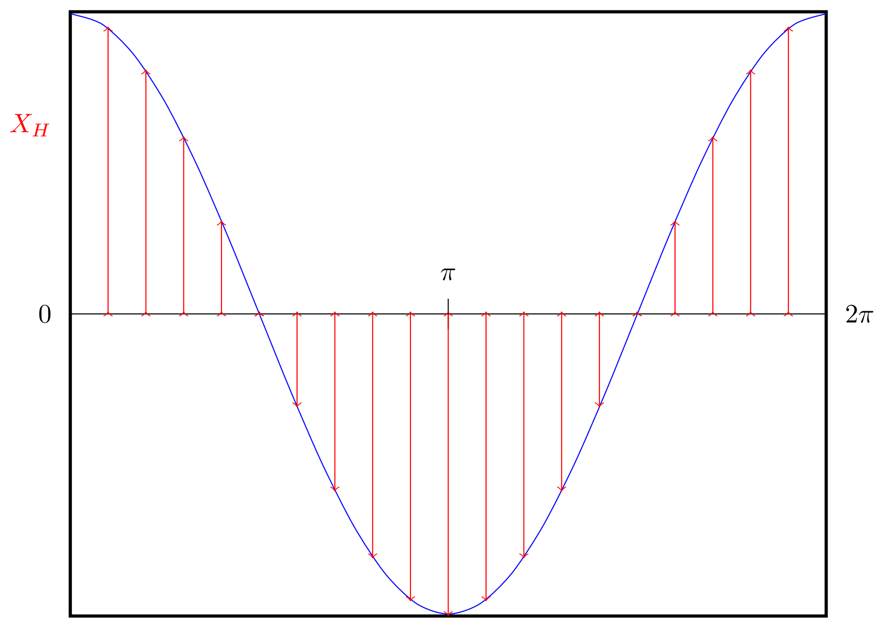
$\square$

[46] There exists a hamiltonian diffeomorphism $\tilde{\psi} : M \to M$ such that $\tilde{\psi}(L) = \text{graph}(\psi)$.

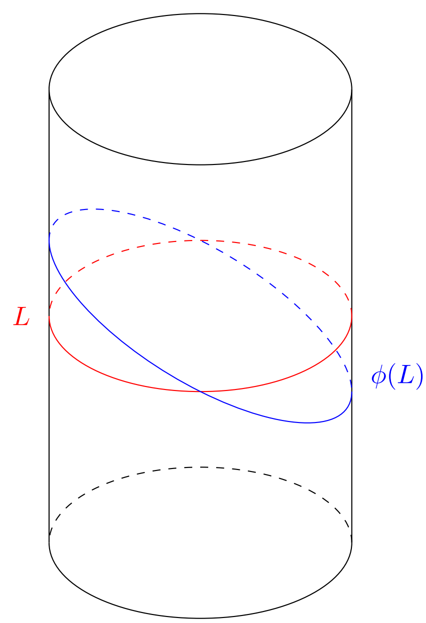
Example. in Floer's context, let $M = \mathbb{R} \times S^1 = \{(x, \theta)\}$, with the symplectic form $\omega = dx \wedge d\theta$.



Then we have a Lagrangian submanifold $L = \{(0, \theta)\}$. If we choose the Hamiltonian function $H = \sin\theta$ then we have $dH = \cos\theta d\theta$, and $X_H = \cos\theta \frac{\partial}{\partial\theta}$ as in the image.



We also then have the symplectomorphism ϕ, with, in particular, $\phi(L) = \{(\cos\theta, \theta)\}$.



So that we see

$$\#(\phi(L) \cap L) = 2 \geq \dim H^*(S^1, \mathbb{Z}/2\mathbb{Z})$$

Floer's Approach: Associate to $(L, \phi(L))$ a cochain complex $CF(L, \phi(L))$ freely generated over Λ by $L \cap \phi(L)$ such that

$$H^*(CF(L, \phi(L))) = H^*(L, \Lambda)$$

Floer's Complex

In what follows, let L_0 and L_1 be compact Lagrangians in a symplectic manifold (M, ω) such that L_0, L_1 intersect transversely.

Definition. The *Novikov Field* Λ_k with base field k is

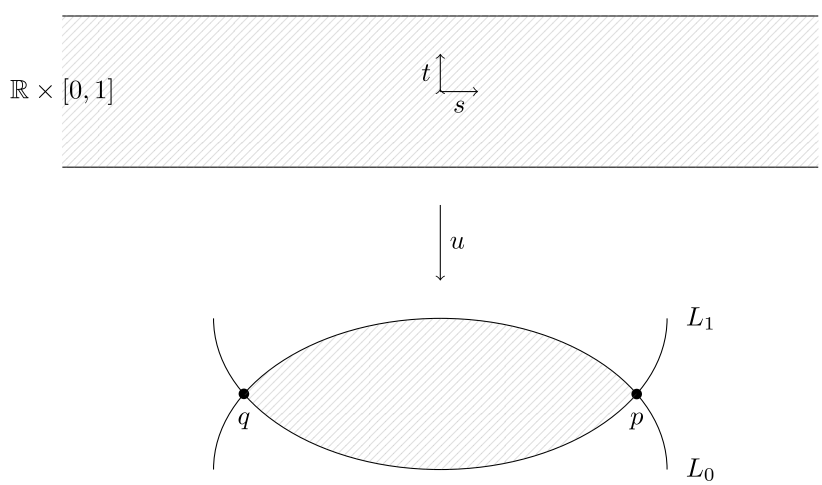
$$\Lambda_k = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \mid a_i \in k,\ \lambda_i \in \mathbb{R},\ \lim_{i \to \infty} \lambda_i = \infty \right\}$$

We can then define the Floer complex as a Λ_k-vecotr space:

$$CF(L_0, L_1) = \bigoplus_{p \in L_0 \cap L_1} \Lambda_k \cdot p$$

To define the differential ∂ we equip M with an ω-compatible almost-complex structure J[47]. The coefficient of q in $\partial(p)$ is given by counting certain pseudo-holomorphic strips with boundary in $L_0 \cup L_1$.

[47] Recall from the last talk that this is a contractible choice.



That is, maps $u : \mathbb{R} \times [0, 1] \to M$ such that

1. (Cauchy-Riemann)
$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} = 0$$

2. (Lagrange Boundary Conditions) $u(s,0) \in L_0$, $u(s,1) \in L_1$ and
$$\lim_{s\to\infty} u(s,t) = p, \quad \lim_{s\to-\infty} u(s,t) = -q$$

3. (Energy Bound)
$$E(u) = \int u^*\omega = \int\int |\frac{\partial u}{\partial s}|^2 \, ds \, dt < \infty$$

The definition of the differential depends on

FUNDAMENTAL RESULT (GROMOV): The space of solutions of (1) (2), and (3)[48] corresponding to a fixed class $[u] \in \pi_2(M, L_1 \cup L_2)$, which we will denote by $\hat{\mathcal{M}}(p,q,[u],J)$, is a smooth manifold whose dimension is given by the maslov index $\mathrm{ind}([u])$. The quotient $\mathcal{M}(p,q,[u],J)$ of $\hat{\mathcal{M}}$ under the $\mathbb{R}$ action given by $u(s,t) \mapsto u(s-a,t)$ is a smooth manifold of dimension $\mathrm{ind}([u]) - 1$.

[48] Technically, a perturbed version of these three equations.

Definition. The *Floer Differential*
$$\partial : CF(L_0, L_1) \to CF(L_0, L_1)$$
is the Λ_k-linear extension of
$$\partial(p) = \sum_{q \in L_0 \cap L_1} \#\mathcal{M}(p,q,[u],J)T^{\omega([u])}q$$
where we abuse notation by letting $[u] = \mathrm{ind}([u]) - 1$[49].

[49] A note on the notation $\#\mathcal{M}$: In general we use $k = \mathbb{F}_2$ and simply take $\#\mathcal{M}$ = parity of sum of points. If L_0, L_1 are oriented and spin, then $\mathcal{M}$ has a natural orientation and $\#\mathcal{M}$ denotes the signed sum of points over any field k.

Floer Homology II

TOBIAS DYCKERHOFF

Last time we defined the Floer complex:

- L_0 and L_1 compact lagrangians in (M, ω) a symplectic manifold.
- L_0 and L_1 intersect transversely.
- We have the Novikov field

$$\Lambda_k = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \mid a_i \in k,\ \lambda_i \in \mathbb{R},\ \lim_{i \to \infty} \lambda_i = \infty \right\}$$

- The FLoer Complex is $CF(L_0, L_1) = \bigoplus_{p \in L_0 \cap L_1} \Lambda \cdot p$ with differential

$$\partial(p) = \sum_{\substack{q \in L_0 \cap L_1 \\ [u] \in \pi_2(M, L_0 \cap L_1) \\ \mathrm{ind}([u])=1}} \#\mathcal{M}(p, q; [u], J) T^{\omega([u])} \cdot q$$

  Where

- J is an ω-compatible almost complex structure.
- $\mathcal{M}(p, q; [u], J)$ is the moduli space of pseudo holomorphic[50] strips of finite energy $\omega([u]) - [u].\omega$ and fixed topological type $[u]$ modulo reparametrization $s \mapsto s - a$.
- $\omega([u])$ is the *energy/symplectic area*
- $\mathrm{ind}([u])$ is the *Maslov index.*

We then have that for all solutions $u \in \mathcal{M}(p, q; [u], J)$ that are *regular*, then[51]

$$\dim \mathcal{M}(p, q; [u], J) = \mathrm{ind}([u]) - 1$$

Maslov Index

STEP 1: Consider the Grassmanian of Lagrangian subspaces $\mathrm{LGr}(2n)$ inside the symplectic vector space

$$(\mathbb{C}^n, dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n)$$

[50] In the sense that $\overline{\partial}_J u = 0$.

[51] This follows from a Riemann-Roch type theorem for Riemann surfaces with boundary (in our case, the strip $\mathbb{R} \times [0, 1]$) and Lagrangian boundary conditions. This yields:

$$\begin{aligned} \mathrm{ind}(D_{\overline{\partial}, u}) &= \dim \ker(D_{\overline{\partial}, u}) - \dim \mathrm{coker}(D_{\overline{\partial}, u}) \\ &= \mathrm{ind}([u]) - 1 \end{aligned}$$

The condition that u be *regular* means that $\dim \mathrm{coker}(D_{\overline{\partial}, u}) = 0$, yielding the conclusion in the text.

where $z_j = x_j + iy_j$. We have that[52]

$$\mathrm{LGr}(2n) \cong U(n)/O(n)$$

[52] This follows from the fact that there is a transitive action of $U(n)$ on $\mathrm{LGr}(2n)$, along with an explicit computation of the kernel.

and therefore we have a map

$$\det{}^2 : U(n)/O(n) \to S^1$$

inducing an isomorphism on fundamental groups. Given a loop ℓ in $\mathrm{LGr}(2n)$, we define its maslov index to be the winding number under the map $\det^2$.

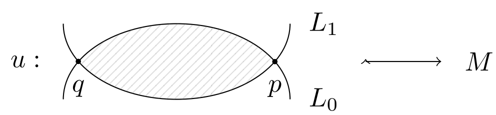
Step 2: Given $\lambda_0, \lambda_1 \in \mathrm{LGr}(2n)$ transverse, then there exists $A \in Sp(2n, \mathbb{R})$ such that $A(\lambda_0) = \mathbb{R}^n$ and $A(\lambda_1) = i\mathbb{R}^n$. Therefore, we obtain a path in $LG(2n)$ from λ_0 to λ_1 given by[53]

$$A^{-1}\left(\exp\left(\frac{-i\pi t}{2}\right)\mathbb{R}\right) \qquad t \in [0,1]$$

[53] One can check that the homotopy class of this path is independant of A

call this the *canonical short path from λ_0 to λ_1*.

Given



we trivialize u^*TM as a symplectic bundle so that we have an isomorphism to $D \times (\mathbb{C}^n, \omega)$. We then get a loop in $\mathrm{LGr}(2n)$[54]:

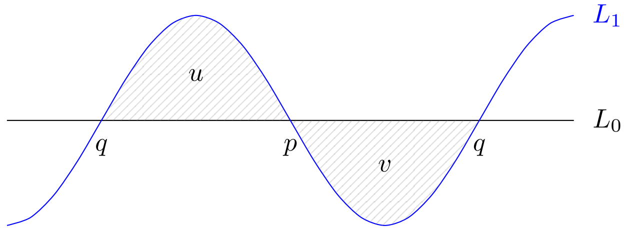
[54] Where we denote the canonical short path by CSP.

$$\begin{tikzcd}
T_qL_1 \arrow[r, rightsquigarrow] & T_pL_1 \arrow[d, rightsquigarrow, "(CSP)^{-1}"] \\
T_qL_0 \arrow[u, rightsquigarrow, "CSP"] \arrow[ur, phantom, "\ell"] & T_pL_0 \arrow[l, rightsquigarrow]
\end{tikzcd}$$

We can then define $\mathrm{ind}([u])$ to be the Maslov index of the loop L.

Example. As in the previous talk, let $M = \mathbb{R} \times S^1$ and $\omega = dx \wedge d\theta$ with the Hamiltonian function $H = \sin\theta$. Then $\psi_H(L_0) = \mathrm{graph}(\cos\theta) =: L_1$. We then have two possible pseudo holomorphic strips[55]:

[55] We can heuristically explain why there is only one possible disk in each gap since $PSL(2, \mathbb{R})$ is dimension 3. Fixing the two points removes two degrees of freedom, and modding out by $s \mapsto s - a$ removes another.



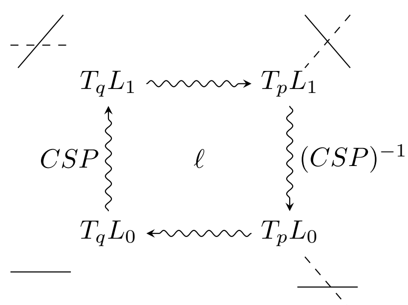
We then have precisely two paths p to q[56] and so we can compute the differential on

$$CF(L_0, L_1) = \Lambda \cdot p \oplus \Lambda \cdot q$$

To do so, however, we mus first compute the maslov index of $[u]$. We have that

$$\mathrm{LGr}(2) = U(1)/O(1) = S^1/\pm 1 = \mathbb{R}P^1$$

and a quick computation shows us



that is, the loop ℓ gives a half-twist. We then have that the winding number of $\det^2(\ell)$ is $1 = \mathrm{ind}([u])$. The differential then becomes:

$$\partial(p) = T^{\omega([u])} \cdot q + T^{\omega([v])} \cdot q$$

However, in $\Lambda_{\mathbb{F}_2}$, the term $T^{\omega([u])} + T^{\omega([v])}$ is precisely zero, and, since we have no strips q to p, we also have $\partial(q) = 0$. S our complex becomes[57]:

$$\Lambda \cdot p \xrightarrow{0} \Lambda \cdot q \;\; \cong \;\; H^*(L_0, \Lambda)$$

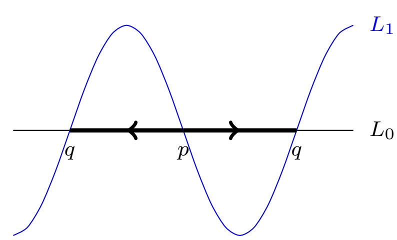
as expected[58].

Example (Non-example)**.** We again take our symplectic manifold to be $M = \mathbb{R} \times S^1$, with form $\omega = dx \wedge d\theta$. However, we now consider a completely different Lagrangian L_1, of the form of a homotopically trivial circle embedded in the side of the cylinder as in the picture

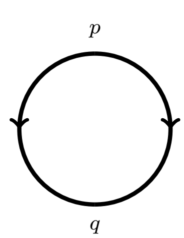
[56] The requirements that both strips be oriented and go from L_0 to L_1 means that both paths must go from p to q.

[57] It is worth noting, that since we have not yet defined a grading on $CF(L_0, L_1)$, we can really only state this for ungraded complexes.

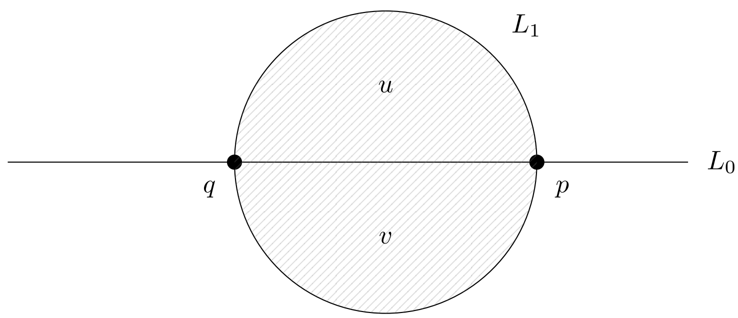
[58] Moreover, we can see that, in this case, we can associate our picture:



To the picture



So that we can see, at least heuristically, that $CF(L_0, L_1)$ is the Morse complex of L_0.



We again have

$$CF(L_0, L_1) = \Lambda \cdot p \oplus \Lambda \cdot q$$

but now

$$\begin{aligned} \partial(p) &= T^2 q \\ \partial(q) &= T^2 p \end{aligned}$$

so that the differential squares to $\partial^2(-) = T^4 \cdot (-)$ rather than zero[59].

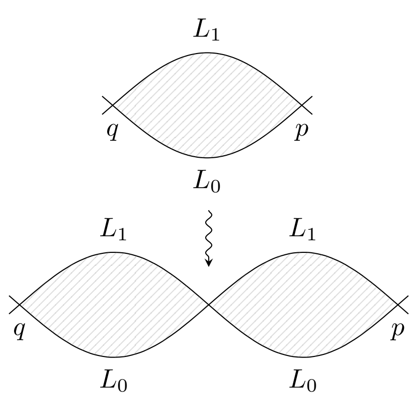
[59] This is why we impose the condition that all disks with boundary in $L_1 \cup L_0$ have trivial symplectic area. Alternately, we could accept $\partial^2 \neq 0$, and get an object called a *twisted* A_∞ *category*.

When/Why $\partial^2 = 0$

We let $[u]$ be a strip with $\mathrm{ind}([u]) = 2$ and study $\mathcal{M}(q, p; [u], J)$, which turns out to be non-compact of dimension 1.

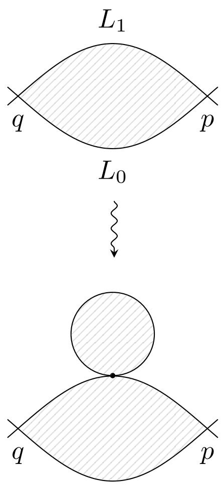
Gromov: $\mathcal{M}(q, p; [u], J)$ can be compactified by adding boundary points corresponding to 3 types of 'degenerate' pseudo-holomorphic strips:

I) *Broken Strips*:

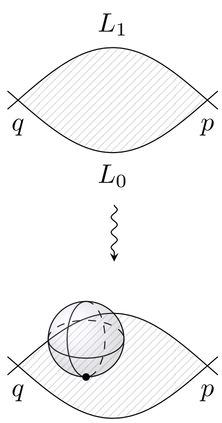


II) *Disk Bubbles*[60]:

[60] One can find explicitly a sequence of strips with energy becoming increasingly concentrated on the boundary to yield such bubbles. See, eg: Auroux,*A Beginners Introduction to Fukaya Categories*.



III) *Sphere Bubbles*:



If II) and III) can be excluded[61], then we have:

$$\partial\overline{\mathcal{M}}(q,p;[u],J) = \bigsqcup_{\substack{r\in L_0\cap L_1\\ [u']+[u'']=[u]\\ \mathrm{ind}[u']=\mathrm{ind}[u'']}} \mathcal{M}(q,r;[u'],J)\times\mathcal{M}(r,p;[u''],J)$$

[61] For example, assume $\pi_2(M,L_i)\cdot\omega = 0$ and $\pi_2(M)\cdot\omega = 0$.

And we have Gromov's compactness theorem:

$$\sum\left[\#\left(\mathcal{M}(q,r;[u'],J)\right)\#\left(\mathcal{M}(r,p;[u''],J)\right)\right]\cdot T^{\omega([u'])+\omega([u''])} = 0$$

The right hand term is precisely the $T^{\omega([u])}$ coefficient of q in $\partial^2(p)$. So we see that, under the assumption that II) and III) are excluded,

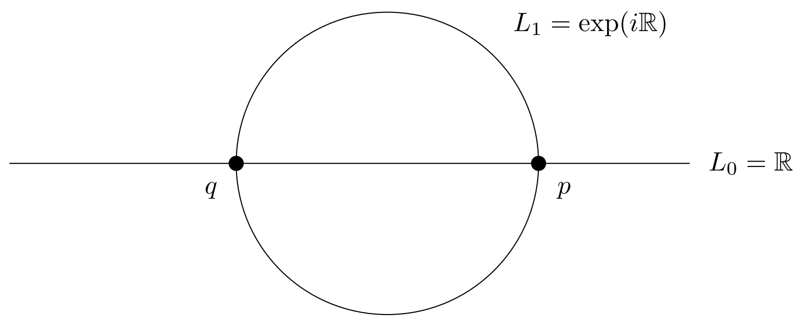
$$\partial^2 = 0$$

The Fukaya Category

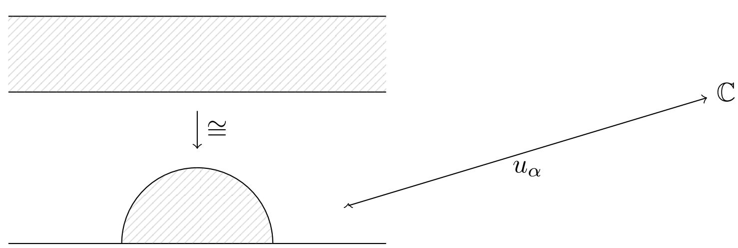
Tobias Dyckerhoff

Continuing from last time, we are interested in the question: Are there any broken strips that appear in surfaces?

Example. Let $M = \mathbb{C}$ with symplectic form $\omega = dx \wedge dy$, and lagrangians as pictured below:



We will explicitly produce a family of pseudo-holomorphic strips in $\mathcal{M}(p, p; [u], J)$ which converge to a broken strip.

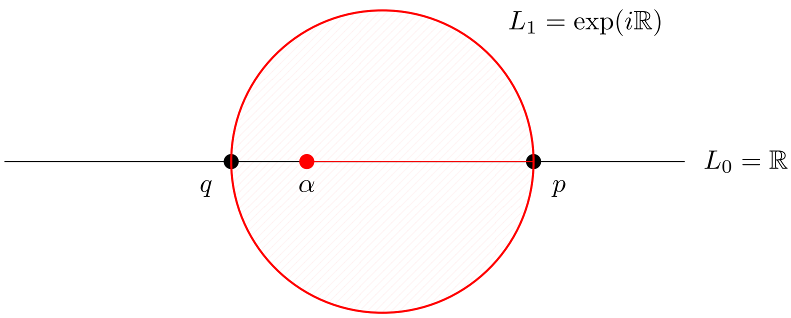


Where the first map is biholomrphic by the Riemann mapping theorem, and the maps u_α depending on a real parameter α are given by:

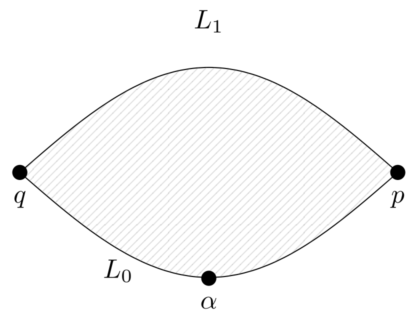
$$u_\alpha(z) = \frac{z^2 + \alpha}{1 + \alpha z^2}$$

which looks like[62]

[62] Note that the topological type of u_α is that of the disk, so that $\text{ind}([u_\alpha]) = 2$



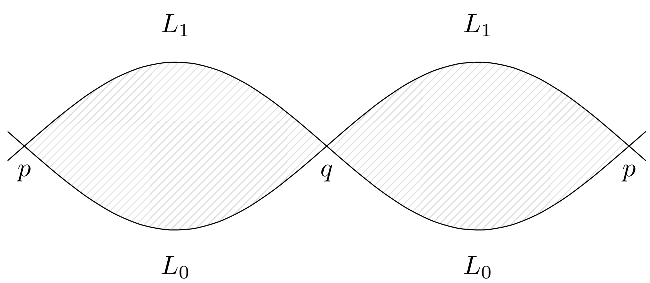
Where, in our diagram of the pseudo-holomorphic strip, we have:



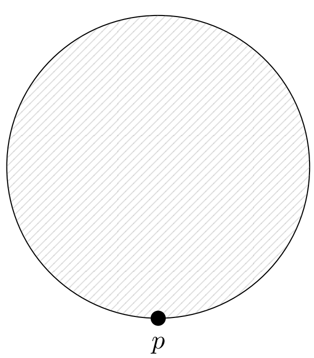
After factoring out reparameterizations, we have

$$\mathcal{M}(p, p; [u], J) \cong (-1, 1) = \{\alpha\}$$

We then have the following degenerations: as $\alpha \to -1$,



And, as $\alpha \to 1$, we get a disk bubble (type II from the previous talk), in which the bottom strip collapses to the point p, leaving:

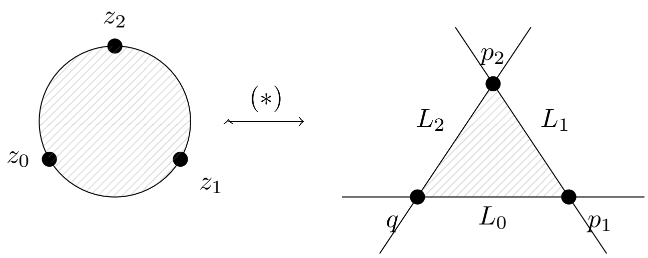


Product Structures

We now move on to trying to define the A_∞-structure on the fukaya category. Let L_0, L_1, L_2 be Lagrangian submanifolds[63] of (M, ω) a symplectic manifold. Then there is a composition

$$CF(L_1, L_2) \otimes CF(L_0, L_1) \to CF(L_0, L_2)$$

given by counting pseudo-holomorphic disks with 3 omitted boundary points[64]:



That is, by counting pseudo-holomorphic maps of D, where D is a disk with labeled points z_0, z_1, z_2 along with a choice of conformal structure up to biholomorphic isomorphism. More precisely,

$$p_2 \cdot p_1 = \sum_{\substack{q \in L_0 \cap L_2 \\ \text{ind}([u])}} \#\mathcal{M}(p_1, p_2, q; [u], J) T^{\omega([u])} \cdot q$$

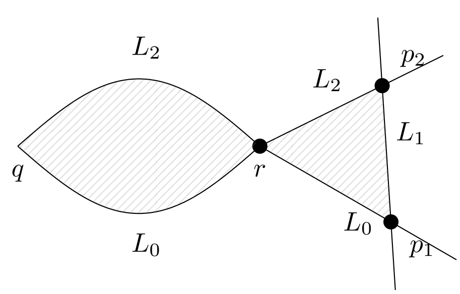
where $\mathcal{M}(p_1, p_2, q; [u], J)$ is the moduli space of disks like $(*)$.

Claim. If $[\omega] \cdot \pi_2(M; L) = 0$ and $[\omega] \cdot \pi_2(M)$ then we have

$$\partial(p_2 \cdot p_1) = \partial(p_2) \cdot p_1 + p_2 \cdot \partial(p_1)$$

The proof follows the same basic idea as the proof that $\partial^2 = 0$. We consider the moduli space $\mathcal{M}(p_1, p_2, q; [u], J)$ with $\text{ind}([u]) = 1$ which is a non compact 1 dimensional manifold. By Gromov, this can be compactified using as boundary points:

I) points corresponding to



II) Diagrams of the same sort, but at the vertex p_2

[63] We require that these intersect transversely, though we permit 3-fold intersections. In general, we can perturb the Cauchy-Riemann equations defining pseudo-holomorphicity to get transversality.

[64] It does not matter where the points are embedded, since the automorphism group $PSL(2, \mathbb{R})$ acts 3-fold transitively on the upper half-place.

III) Diagrams of the same sort, but at the vertex p_1

We then notice that counting the number of type I) gives us the $\partial(p_2 \cdot p_1)$, type II) gives use $\partial(p_2) \cdot p_1$ and type III) gives use $p_2 \cdot \partial(p_1)$. Since the boundary points come in pairs[65] we get that the formula holds modulo 2.

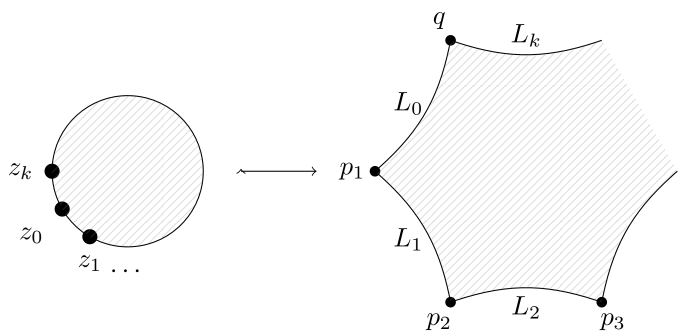
[65] A compact 1-manifold with boundary is simply a union of intervals and circles, so it has an even number of boundary components. Gromov's compactness theorem states that there is a bijection between all possible degeneracies of types (I) (II) and (III), and the boundary points of the compactification of $\mathcal{M}(p_1, p_2, q; [u]J)$.

Higher Operations

Given a set of transverse Lagrangians $L_0, \ldots, L_k$ in (M, ω). we can define

$$\mu_k : CF(L_{k-1}, L_k) \otimes \cdots \otimes CF(L_0, L_1) \to CF(L_0, L_k)$$

given by counting pseudo-holomorphic disks of the sort:



up to the action of

$$\mathrm{Aut}(\mathbb{D}) \cong \mathrm{PSL}(2, \mathbb{R})$$

or, more precisely:

$$\mu_k(p_k, \ldots, p_1) = \sum \#\mathcal{M}(p_1, \ldots, p_k, q; [u], J) \cdot T^{\omega([u])} q$$

where the sum ranges over $q \in L_0 \cap L_1$ and $\mathrm{ind}([u]) = 2 - k$[66].

[66] This value is chosen because, assuming regularity, we have

$$\dim \mathcal{M}(p_1, \ldots, p_k, q; [u], J) = (k+1) - 3 + \mathrm{ind}([u])$$

The formulas relating the various operations μ_k come from considering compactifications of the moduli spaces listed above. The boundary points in these compactifications correspond to

1. Strip breaking.

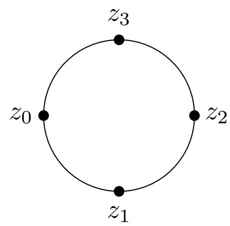
2. collisions among the label set $\{z_0, z_1, \ldots, z_k\}$.

As a warm up, consider $\mathcal{M}_{0,n+1}$, the moduli space of conformal structures on the disk with $n + 1$ chosen boundary points,which is equivalent to

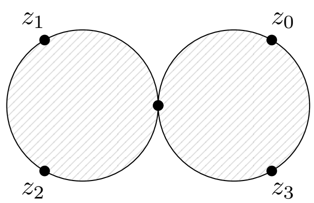
$$\left\{ \begin{array}{l} \text{Labeled configurations} \\ \text{of points on} \\ S^1 \text{ the unit circle} \end{array} \right\}_{/\mathrm{PSL}(2,\mathbb{R})}$$

For example, we have

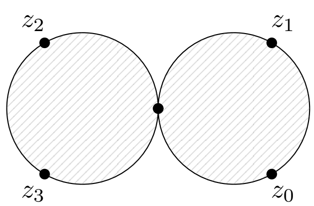
$\mathcal{M}_{0,4} \quad =$



Since $\mathrm{PSL}(2,\mathbb{R})$ can send three points to any other three points, we can restrict oursives to considering the motion of z_3, so that we get an open interval. The compactification results in our consider what happens when z_3 collides with one of the other points. If z_3 collides with z_0, we compactify by adding the moduli of



and, when z_3 collides with z_2, we use



In each case, this amounts to adding a point, so that we get the closed interval.

Exercise. $\overline{\mathcal{M}}_{0,5} = K_4$ the fourth stasheff polytope[67].

The A_∞ relation[68]

$$\partial \circ m_3 + m_3 \circ \partial = m_2(m_2 \times 1) + m_2(1 \times m_2)$$

follows from counting the boundary of

$$\overline{\mathcal{M}}(p_1, p_2, p_3, q; [u], J)$$

The left hand side of the relation corresponds to strip breaking, whereas the right hand side corresponds to collisions.

Fact. With sufficient care, one can show that the A_∞ relations

$$\sum_{\ell=1}^{k} \sum_{j=0}^{k-\ell} \mu^{k+1-\ell}(p_k, \ldots, p_{k+\ell+1}, \mu^{\ell}(p_{j+\ell}, \ldots, p_{j+1}), p_j, \ldots, p_1) = 0$$

hold.

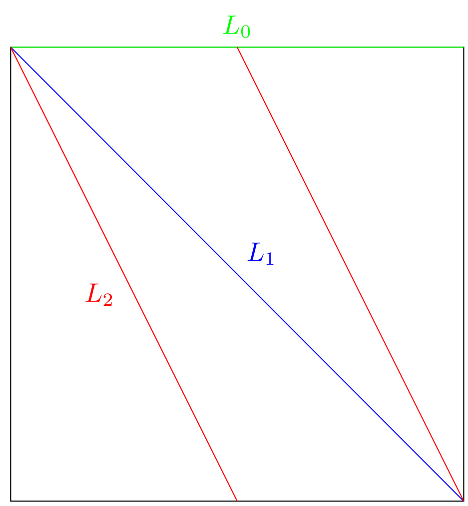
[67] More generally, it is true that $\overline{\mathcal{M}}_{0,n+1} = K_n$.

[68] Where, as previously, $m_3 \circ \partial = m_3(\partial \times 1 \times 1) + m_3(1 \times \partial \times 1) + m_3(1 \times 1 \times \partial)$.

Example. Let $M = \mathbb{R}^2/\mathbb{Z} \oplus \mathbb{Z}$ and $\omega = \lambda dx \wedge dy$ for $\lambda \in \mathbb{R}_{>0}$. And consider Lagrangians given by

$$L_0 = \mathbb{R} \cdot (1,0) \quad L_1 = \mathbb{R} \cdot (1,-1) \quad L_2 = \mathbb{R} \cdot (1,-2)$$

as in the diagram



We then have

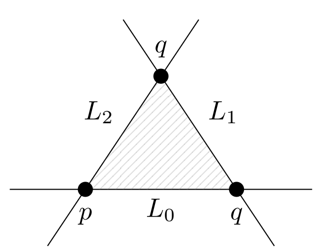
$$CF(L_0, L_{-1}) = \Lambda \cdot q \quad CF(L_1, L_2) = \Lambda \cdot q \quad CF(L_0, L_2) = \Lambda \cdot p \oplus \Lambda \cdot q$$

And pair of langrangians diverge from each other, so we have no pseudo-holomorphic strips, and thus $\partial = 0$ everywhere.

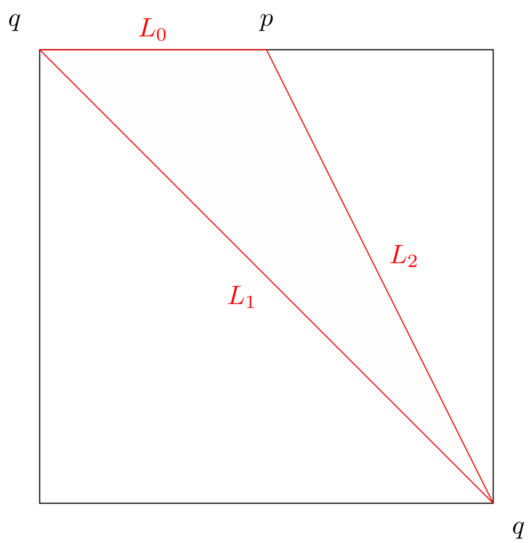
We can then compute the map

$$CF(L_1, L_2) \otimes CF(L_0, L_1) \to CF(L_1, L_2)$$

We write $m_2(q, q) = A \cdot p + B \cdot q$ and compute A and B. To compute A, we must count triangles of the shape



So we find (in the universal cover)



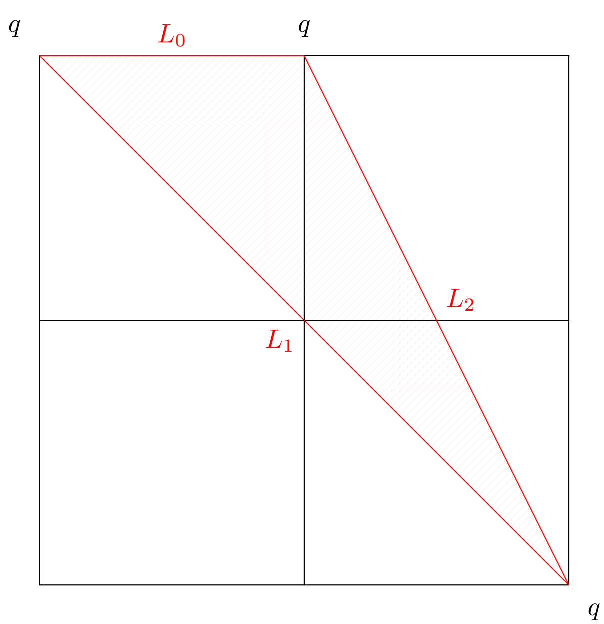
So we must count[69] triangles with vertices

$$\left\{(0,0),(0,n+\frac{1}{2}),(2n+1,-(2n+1))\right\}$$

which have area $\lambda(n+\frac{1}{2})^2$. So that

$$A=\sum_{n\in\mathbb{Z}}T^{\lambda(n+\frac{1}{2})^2}$$

We can also do the same for B:



so that we conside triangles with vertices

$$\{(0,0),(n,0),(2n,-2n)\}$$

[69] It requires some additional justification, omitted here, to see why we simply count immersed triangles up to reparametrization.

yielding

$$B = \sum_{n\in\mathbb{Z}} T^{\lambda n^2}$$

so that

$$m_2(q,q) = \sum_{n\in\mathbb{Z}} T^{\lambda(n+\frac{1}{2})^2} \cdot p + \sum_{n\in\mathbb{Z}} T^{\lambda n^2} \cdot q$$

Example (Mirror Symmetry)**.** From mirror symmetry we have

$$\mathrm{Fuk}(\mathbb{R}^2/\mathbb{Z}^2, \lambda dx \wedge dy) \cong D^b(\mathrm{Coh}, \mathbb{C}/\mathbb{Z} \oplus i\lambda)$$

Where Fuk is the Fukaya category and Coh stands for coherent sheaves. We then have correspondences

$$\begin{array}{lcl} L_0 & \sim & \mathcal{O} \\ L_1 & \sim & \begin{array}{l}\text{Line bundle of degree 1} \\ \text{Take } \mathcal{O}(P) \text{ for } P \in E\end{array} \\ L_2 & \sim & \mathcal{O}(2P) \end{array}$$

We can then compute:

$$\begin{array}{rcl} CF(L_0, L_1) & = & \mathrm{Hom}(\mathcal{O}, \mathcal{O}(P)) \quad \dim = \deg = 1 \\ CF(L_1, L_2) & = & \mathrm{Hom}(\mathcal{O}(P), \mathcal{O}(2P)) \quad \dim = 1 \\ CF(L_0, L_2) & = & \mathrm{Hom}(\mathcal{O}, \mathcal{O}(2P)) \quad \dim = 2 \end{array}$$

Part III

Applications

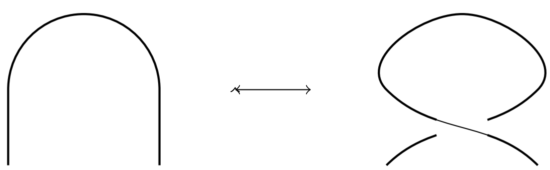
Categorifying the Jones Polynomial

Walker Stern

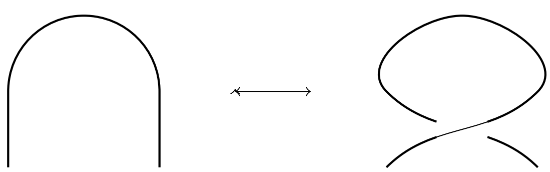
Definition. A *link* L is an isotopy class of embeddings of circles into $\mathbb{R}^3$. A *plane projection* D of a link L is a projection of L onto a plane in $\mathbb{R}^3$ remembering over/under intersections. An isotopy class of plane projections is called a *plane diagram* of L. A plane diagram D is called *generic* if it has no triple intersections, cusps, or tangencies.

The conditions under which two plane diagrams represent the same link are well-known. In particular, D_1 and D_2 represent the same link if and only if they are related by the *Reidemeister Moves:*

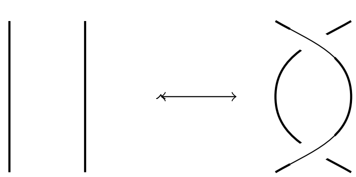
I. Left twist



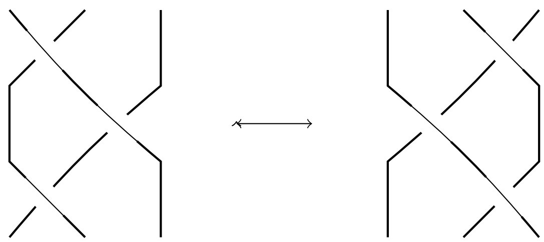
II. Right twist



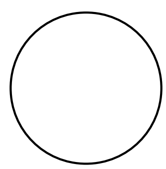
III. Tangency

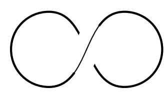


IV. Triple Point



Example. Making use of the left twist we find that the diagrams





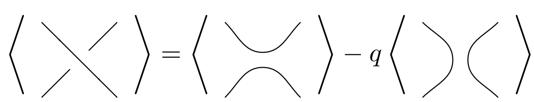
both represent the unknot.

To a plane diagram D it is possible to associate a polynomial

$$\langle D \rangle \in \mathbb{Z}[q, q^{-1}]$$

according to the rules

1. $\langle \bigcirc \rangle = q + q^{-1}$, where $\bigcirc$ represents the unknot.
2. For links only differing inside the pictured area,



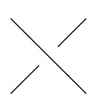
3. For two diagrams D_1 and D_2,

$$\langle D_1 \sqcup D_2 \rangle = \langle D_1 \rangle \langle D_2 \rangle$$

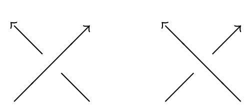
It is important to note that $\langle D \rangle$ is **not** invariant under Reidemeister moves[70].

[70] The only move under which it is invariant is the triple point move. It is a quick calculation to see that the right twist introduces a factor of $-q^{-2}$, the left twist a factor of q, and the tangency move a factor of $-q^{-1}$.

However, in the definition of this polynomial, we have neglected the orientation of links. In a diagram of an oriented link, the crossings represented above as



can instead be represented as either of



Given a diagram D, we will call the number of crossings of the first kind $y(D)$ and the number of crossings of the second kind $x(D)$.

Definition. The *Scaled Kauffman Bracket*[71] is an invariant of a link L gievn by

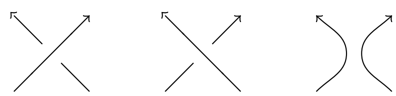
$$K(L) = (-1)^{x(D)} q^{y(D)-2x(D)} \langle D \rangle$$

for any diagram D of L.

There is another, closely related link invariant:

Definition. The *Jones Polynomial* $V(L)$ of an oriented link is determined by two properties:

1. The Jones Polynomial of the unknot is 1
2. For L_1, L_2, L_3 links differing only by



L_1 L_2 L_3

we have

$$t^{(} - 1)V(L_1) - tV(L_2) = (\sqrt{t} - \frac{1}{\sqrt{t}})V(L_3)$$

It is a well-known fact[72] that

$$V(L)_{\sqrt{t}=-q} = \frac{K(L)}{q+q^{-1}}$$

Algebraic Preparations

We will need some fixed algebraic objects for the rest of the talk.

- Let $R = \mathbb{Z}[c]$ be a $\mathbb{Z}$-graded ring, with the grading given by

  $$\deg(1) = 0 \quad \deg(c) = 2$$

  Let $R - \mathrm{mod}_0$ be the abelian category of graded R-modules, with objects $M = \bigoplus M_i$ and morphisms graded maps of degree 0. For $n \in \mathbb{Z}$ we have and automorphism of $R - \mathrm{mod}_0$, denoted $\{n\}$, and given by

  $$M\{n\}_i = M_{i+n}$$

- Let A be the free rank 2 R-module spanned by $\mathbf{1}$ and $\mathbf{X}$ such that

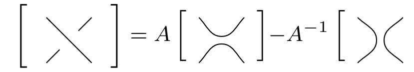
  $$\deg(\mathbf{1}) = 1 \quad \deg(\mathbf{X}) = -1$$

  The multiplication is given by $\mathbf{1X} = \mathbf{X1} = 1$ and $\mathbf{X}^2 = 0$. The unit is the map $i : R \to A$ given by $1 \mapsto \mathbf{1}$

  A also has a coalgebra structure, given by morphisms Δ and ϵ (comultiplication and counit)[73]

[71] More conventional is the Kauffman bracket, $f[L]$, for which the polynomial $[D] \in \mathbb{Z}[A, A^{-1}]$ is given by the rules

1. $[\bigcirc] = 1$
2. For diagrams differing only inside the pictured area:



3. $[\bigcirc \sqcup D] = (-A^2 - A^{-2})[D]$

Under this definition, we have

$K(L)_{(q=-A^2)} = (-A^2 - A^{-2}) f[L]$

[72] This is not difficult to show directly. We have that K of the unknot is $q + q^{-1}$, and one can explicitly compute that for L_1, L_2, and L_3, we have the relation:

$q^{-2}K(L_1) - q^2 K(L_2) = (q^{-}1 - q)K(L_3)$

[73] More explicitly,

$$\begin{array}{rcl} \epsilon : A & \to & R \\ \epsilon(\mathbf{1}) & = & -c \\ \epsilon(\mathbf{X}) & = & 1 \\ \Delta : A & \to & A \otimes A \\ \Delta(\mathbf{1}) & = & \mathbf{1} \otimes \mathbf{X} + \mathbf{X} \otimes \mathbf{1} + c\mathbf{X} \otimes \mathbf{X} \\ \Delta(\mathbf{X}) & = & \mathbf{X} \otimes \mathbf{X} \end{array}$$

Definition. The *graded euler characteristic* of $M \in R - \mathrm{mod}_0$ is

$$\hat{\chi}(M) = \sum_{j \in \mathbb{Z}} \dim_{\mathbb{Q}}(M_j \otimes \mathbb{Q}) q^j$$

We also notice that we have two automorphisms of the category of cochain complexes $Kom(R - \mathrm{mod}_0)$: $[n]$ given by shifting the cochain down, and $\{n\}$ given by shifting the grading.

The final piece of algebraic formalism we will need is the notion of I-cubes. Let I be a finite set, and B a category. let

$$r(I) = \{(\mathcal{L}, a) | \mathcal{L} \subset I, a \in I \setminus \mathcal{L}\}$$

Definition. A *commutative I-cube V* over B consists of

- For each $\mathcal{L} \subset I$ an object $V(\mathcal{L}) \in B$
- For each $(\mathcal{L}, a) \in r(I)$, a morphism in B:

$$\xi_a^V(\mathcal{L}) : V(\mathcal{L}) \to V(\mathcal{L}a)$$

such that for any triple $(\mathcal{L}, a, b)$ with $a \neq b$ and neither lying in $\mathcal{L}$

$$\xi_b^V(\mathcal{L}a) \circ \xi_a^V(\mathcal{L}) = \xi_a^V(\mathcal{L}b) \circ \xi_b^V(\mathcal{L})$$

Examples. • If $I = \emptyset$, V is simply a choice of object in B.

- If $I = \{a\}$, V is a choice of a morphism in B
- If $I = \{a, b\}$, then V is a commutative diagram:

$$\begin{tikzcd}
V(b) \arrow[r] & V(ab) \\
V(\emptyset) \arrow[r] \arrow[u] & V(a) \arrow[u]
\end{tikzcd}$$

More generally, for $|I| = n$, the objects of V will correspond to the vertices of the standard n-cube in euclidean space, and the morphisms will correspond to the edges.

Definition. A *morphism of commutative I-cubes* $V \to W$ is a collection of morphisms in B

$$\psi(\mathcal{L}) : V(\mathcal{L}) \to W(\mathcal{L})$$

such that, for all $(\mathcal{L}, a) \in r(I)$ TFDC

$$\begin{tikzcd}
V(\mathcal{L}) \arrow[r, "\psi(\mathcal{L})"] \arrow[d] & W(\mathcal{L}) \arrow[d] & \\
V(\mathcal{L}a) \arrow[r] & \psi(\mathcal{L}a) & W(\mathcal{L}b)
\end{tikzcd}$$

Fact. If B is an abelian category, commutative I-cubes over B from an abelian category.

Now suppose I is a finite set with $I = J \sqcup \{a\}$. Given a commutative I-cube V, there are 2 ways to extract a J-cube from it[74]

- $V_a(*0)(\mathcal{L}) = V(\mathcal{L})$
- $V_a(*1)(\mathcal{L}) = V(\mathcal{L}a)$

This establishes a 1-1 correspondence between I-cubes and morphisms of J-cubes.

Definition. For I a finite set, B an additive category, a *skew-commutative I-cube* (or *skew I-cube*) over B is an I cube, where, instead of requireing the faces to commute, we require:

$$\xi_b^V(\mathcal{L}a) \circ \xi_a^V(\mathcal{L}) + \xi_a^V(\mathcal{L}b) \circ \xi_b^V(\mathcal{L}) = 0$$

We can define tensor product and direct sum of I-cubes over $R-\text{mod}_0$ objectwise, and we find, in particular, that $V \otimes W$ is

- A commutative I-cube if V and W are both skew or both commutative
- A skew I-cube if V is skew and W is commutative, or vice versa.

This means that we can pass from commutative I-cubes to skew I-cubes by tensoring with a skew I-cube. We now construct a special skew I-cube, whose tensor product action on commutative I-cubes is in some sense the same as inserting signs to make morphisms anticommute.

Given $\mathcal{L}$ a finite set, denote by $o(\mathcal{L})$ the set of total orders of $\mathcal{L}$. For $x, y \in o(\mathcal{L})$, let $p(x,y)$ be the parity of the symmetric group element sending x to y.

Let $E(\mathcal{L})$ be the quotient of the free R-module on $o(\mathcal{L})$ by the relations

$$x = (-1)^{p(x,y)} y$$

for every $x, y \in o(\mathcal{L})$. It is immediate that $E(\mathcal{L})$ is a rank 1 R-module.

We also get maps $E(\mathcal{L}) \to E(\mathcal{L}a)$ associated to the maps

$$\begin{array}{ccc} o(\mathcal{L}) & \to & o(\mathcal{L}) \\ x & \mapsto & xa \end{array}$$

We can show that these maps are such that

$$\begin{tikzcd} E(\mathcal{L}) \arrow[r] \arrow[d] & E(\mathcal{L}a) \arrow[d] \\ E(\mathcal{L}b) \arrow[r] & E(\mathcal{L}ab) \end{tikzcd}$$

[74] For example, let $I = \{a, b\}$, and V the diagram

$$\begin{tikzcd} V(b) \arrow[r] & V(ab) \\ V(\emptyset) \arrow[r] \arrow[u] & V(a) \arrow[u] \end{tikzcd}$$

Then $V_a(*0)$ is the morphism

$$V(\emptyset) \to V(b)$$

and $V_a(*1)$ is the morphism

$$V(a) \to V(ab)$$

anticommutes. So, for any finite set I, we get a skew I-cube E_I given by

$$E_I(\mathcal{L}) = E(\mathcal{L})$$

Definition. Given V a skew I-cube over B an abelian category, we can define the complex[75] of V to be

$$\overline{C}^\bullet(V) = (\overline{C}^i, d^i)_{i\in\mathbb{Z}}$$

with

$$\overline{C}^i(V) = \bigoplus_{\substack{\mathcal{L}\subset I\\ |\mathcal{L}|=i}} V(\mathcal{L})$$

and, for $x \in V(\mathcal{L})$ $d^i(x) = \sum_{a\in I\setminus\mathcal{L}} \xi_a^V(\mathcal{L})x$

[75] For example, let $I = \{a, b\}$, then

$$\overline{C}^\bullet(V) = \begin{cases} V(\emptyset) & i=0\\ V(a)\oplus V(b) & i=1\\ V(ab) & i=2\\ 0 & else\end{cases}$$

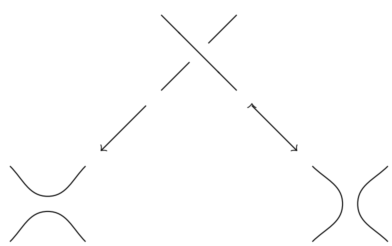
A quick computation of the differentials gives us

$d^1\circ d^0 = \xi_b^V(a)\circ\xi_a^V(\emptyset)+\xi_a^V(B)\circ\xi_b^V(\emptyset) = 0$

From Links to Cubes

Now, we would like a mechanism whereby we can extract an I-cube of some sort from a link. To do this, take a diagram D of L.

A double point of D can be *resolved* in two ways:



0-res 1-res

Definition. A *resolution* of a plane diagram D is a resolution of each double point of D.

Let I be the set of double points of D. There is a bijection between the set of resolutions of D and subsets of I given by

$$\mathcal{L} \mapsto \text{resolution given by 1-res on } \mathcal{L}$$

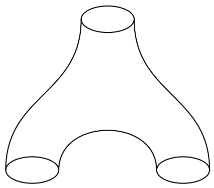
Digression:TFT

Definition. A *topological field theory* (TFT) is a symmetric monoidal functor from a bordism category to another symmetric monoidal category.

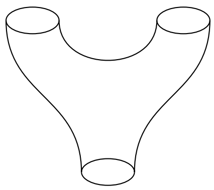
There is a TFT

$$F : 2\mathrm{Cob}^{cl} \to R - \mathrm{mod}_0$$

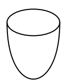
with $F(S^1) = A$ and (where the cobordisms are read from bottom to top)



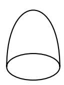
$\mapsto m : A \times A \to A$



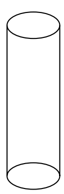
$\mapsto \Delta : A \to A \times A$



$\mapsto \quad i : R \to A$



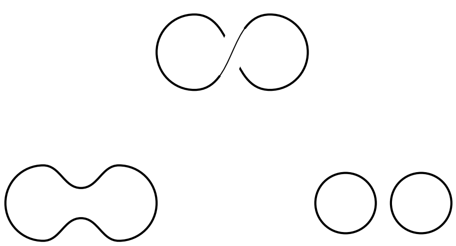
$\mapsto \quad \epsilon : A \to R$



$\mapsto \quad Id_A$

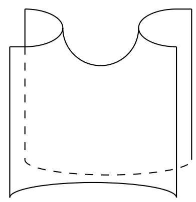
The Complex of a Link

A resolution of D gives a union of k circles, for example:



Now, given 2 resolutions of D differing at a single double point a, $D(\mathcal{L})$ and $D(\mathcal{L}a)$, we can find a neighborhood U of a such that the resolutions differ only inside U. We will use this neighborhood to construct a canonical cobordism $S_a(\mathcal{L})$ between $D(\mathcal{L})$ and $D(\mathcal{L}a)$.

This construction boils down to the requirement that, outside U, the cobordism $S_a(\mathcal{L})$ be the cylinder on $D(\mathcal{L})$, and that inside U it be given by[76]:



[76] More technically, we require that S be the surface embedded in $\mathbb{R}^2 \times [0,1]$ such that

1. $\partial(S) = D(\mathcal{L}) \sqcup D(\mathcal{L}a)$
2. outside $U \times [0,1]$,
$$S = \left[D(\mathcal{L}) \cap \left(\mathbb{R}^2 \setminus U\right)\right] \times [0,1]$$
3. The connected component of S that has non-trivial intersection with $U \times [0,1]$ is homeomorphic to the pair of pants.
4. The projection $S \to [0,1]$ has only one critical point.

It is immediate to observe that compositions of these cobordisms commute in the expected ways, so that we have a commutative I-cube in $2\mathrm{Cob}^{cl}$. We can then use the functor F to define a commutative I-cube in $R - \mathrm{mod}_0$, by

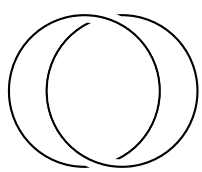
- $V_D(\mathcal{L}) := F(D(\mathcal{L}))\{-|\mathcal{L}|\}$
- $\xi_a^V(\mathcal{L}) := F(S_a(\mathcal{L}))$

An explicit computation shows

$$\deg\left(F(S_a(\mathcal{L}))\right) = -1$$

so that $\xi_a^V(\mathcal{L})$ preserves the shifted gradings.

Example. If we take the link diagram



we get the cube

$$\begin{tikzcd} A^{\otimes 2} \arrow[r, "m"] \arrow[d, "m"'] & A \arrow[d, "\Delta"] \\ A \arrow[r, "\Delta"] & A^{\otimes 2} \end{tikzcd}$$

Now that we have a cube associated to a diagram D, we can extract a complex from it:

Definition. We define

$$\overline{C}^{\bullet}(D) = \overline{C}^{\bullet}(V_D \otimes E_I)$$

and

$$C^{\bullet}(D) = \overline{C}^{\bullet}(D)[x(D)]\{2x(D) - y(D)\}$$

And we can now state the main theorem of this talk:

Theorem. *For an oriented link L,*

$$K(L) = (1 - q^2)\sum_{i\in\mathbb{Z}}(-1)^i\hat{\chi}(H^i(D))$$

where $H^i(D) := H^i(C^\bullet(D))$.

Proof. For a complex $M^\bullet$ in $Kom(R - \mathrm{mod}_0)$, define

$$\hat{\chi}(M^\bullet) := \sum_{i\in\mathbb{Z}}(-1)^i\hat{\chi}(M^i)$$

Then it is a simple fact form homological algebra that

$$\hat{\chi}(C^\bullet(D)) = \sum_{i\in\mathbb{Z}}(-1)^i\hat{\chi}(H^i(D))$$

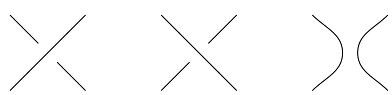
Additionally, we have that, for a graded R-module M,

$$\hat{\chi}(M\{n\}) = q^{-n}\hat{\chi}(M)$$

and for a complex $M^\bullet$

$$c\hat{h}i(M^\bullet[n]) = (-1)^n\hat{\chi}(M^\bullet)$$

Now, let D_1, D_2, D_3 be plane diagrams differing by



D_1 D_2 D_3

The cobordism described above yields a map of complexes

$$\overline{C}^\bullet(D) \xrightarrow{f} \overline{C}^\bullet(D_3)\{-1\}$$

such that $\overline{C}^\bullet(D_1)[1]$ is isomorphic to $\mathrm{cone}(f)$. Hence

$$\hat{\chi}(\overline{C}^\bullet(D_1)) = c\hat{h}i(\overline{C}^\bullet(D_2)) - q\hat{\chi}(\overline{C}^\bullet(D_3))$$

and

$$c\hat{h}i(C^\bullet(D)) = (-1)^{x(D)}q^{y(D)-2x(D)}c\hat{h}i(\overline{C}^\bullet(D))$$

and, on the unknot diagram $\bigcirc$,

$$c\hat{h}i(\overline{C}^\bullet(\bigcirc)) = c\hat{h}i(A) = (q + q^{-1})$$

These conditions correspond precisely to

$$\begin{aligned} \langle D_1\rangle &= \langle D_2\rangle - q\langle D_3\rangle \\ \langle\bigcirc\rangle &= (q + q^{-1}) \\ K(D) &= (-1)^{x(D)}q^{y(D)-2x(D)}\langle D\langle \end{aligned}$$

proving the proposition. $\square$

Khovanov also proves another result, showing that, in some sense, the invariants achieved with this technique are finer than the Jones polynomial.

Theorem. *If D is a plane diagram of an oriented link L, then for $i \in \mathbb{Z}$, the isomorphism class of graded R-modules $H^i(D)$ is an invariant of L.*

The theorem is proved by explicitly constructing a quasi-isomorphism for each Reidemeister move, by means of a cobordism between link diagrams.

Knot Invariants

Catharina Stroppel

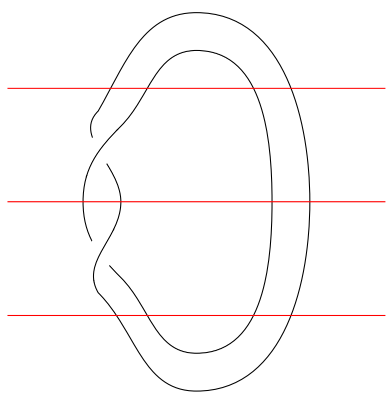
Khovanov Homology

Given a diagram D of a knot or link L, we can use Khovanov's machinery to get a bigraded vector space

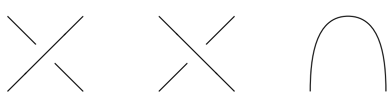
$$\bigoplus_{i,j} H^{i,j}(D)$$

which is an invariant of L.

For a more general setup, we can cut a knot or link L into 'braids' as in the diagram.



we get from this the *elementary pieces*



Consider the Lie Algebra $\mathfrak{sl}_k(\mathbb{C})$, which consists of traceless matrices. We have an inclusion

$$\mathfrak{sl}_k \hookrightarrow \mathbb{C}^k =: V$$

which in turn gives

$$\mathfrak{sl}_k \hookrightarrow V^{\otimes n} \circlearrowleft \mathbb{C}[S_n]$$

which is injective for some k. Ideally, we would like a braid group action instead of a S_n-action[77].

[77] This would allow us to, in some sense, distinguish between the elementary pieces. A symmetric group action identifies the first two.

So how do we get a braid group action? They arise naturally from the Yang-Baxter equations, which leads us to the study of quantum groups. As an analogue of our inclusions above, in this case we get

$$U_q(\mathfrak{sl}_k(\mathbb{C})) \hookrightarrow (V_q)^{\otimes n} \circlearrowleft \mathbb{C}_q[S_n]$$

where $V_q := \mathbb{C}(q)$. We denote the Hecke Algebra $\mathbb{C}_q[S_n]$ by $\mathcal{H}(S_n)$. Letting I_k be the kernel of the action of $\mathcal{H}_q(S_n)$, we can take the quotient[78]

$$\mathbb{C}_q[S_n]/I_k$$

Example. In the case $k = 2$, $n \geq 2$,

$$\mathcal{H}_q(S_n)/I_k = \mathcal{H}_q(S_n)/\left\langle \sum_{\omega \in S_2} qT_\omega \right\rangle$$

Presentation: We can come up with new generators given by

$$C_i := T_i + q$$

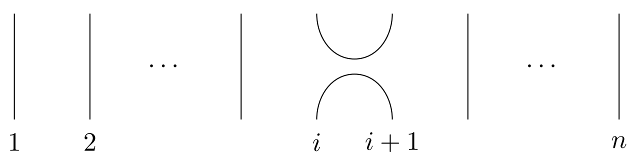
which will then satisfy relations

$$C_i^2 = (q + q^-1)C_i = [2]C_i$$

where [2] is the q-analogue[79].

$$\begin{array}{rcll} C_iC_j & = & C_jC_i & |i-j| > 1 \\ C_iC_jC_i & = & C_i & |i-j| = 1 \end{array}$$

An we have a correspondence between these generators and the diagrams

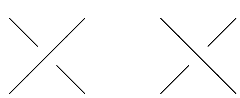
$C_i \longleftrightarrow$ 

And can assign to the cup and the cap the (co)unit

$$\begin{array}{rcl} \cap : V^* \otimes V & \to & \mathbb{C}(q) \\ \cup : \mathbb{C}(q) & \to & V \otimes V^* \\ 1 & \mapsto & v_0 \otimes v_1 + qv_1 \otimes v_0 \end{array}$$

where V is generated by v_0 and v_1.

If we set



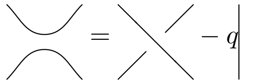
T_i^{-1} $\qquad$ T_i

[78] Braid relations

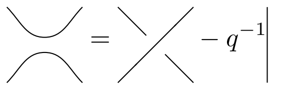
$$T_i^2 = 1 + (q^-1 - q)T_i$$

[79] Note that, in general specializing to $q = 1$ will yield a factor of k in this position.

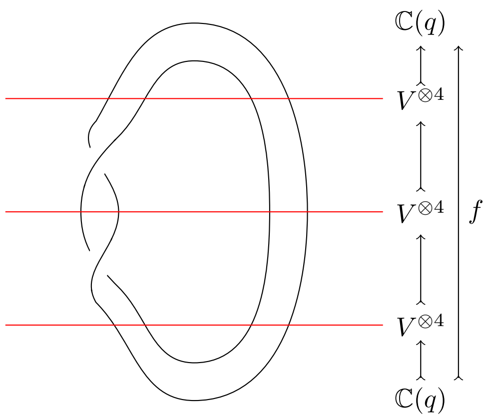
then we get the *skein relations*:



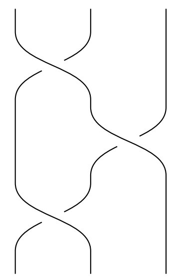
and



So, how do we now go about finding invariants for a link/braid? In our diagram from the beginning[80], we can see that the assignments from the example give us a morphism f:



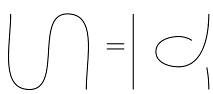
[80] We can do much the same for a braid, for instance, assigning to the braid



a morphism $V^{\otimes 3} \to V^{\otimes 3}$.

In this case, we find that $f(1)$ is simply the Jones polynomial.

To do this more generally we a category with additional structure[81], for example, relations like:



[81] More precisely, we need a braided monoidal category with left and right duals that are canonically isomorphic, and some move invariances.

Example. We can compute invariants of the unknot. By breaking it into a cup and a cap, we get a morphism

$$\begin{array}{rcl} \mathbb{C}(q) & \to & V \otimes V \to C(q) \\ 1 & \mapsto & qv_1 \otimes v_0 + v_0 \otimes v_1 \mapsto [2] \end{array}$$

So that 1 is sent to the quantum dimension of V[82].

[82] Just as

$$2 = \dim \mathbb{C}[X]/(X^2)$$

We have

$$[2] = q + q^{-1} = \mathrm{qdim}(\mathbb{C}[X]/(X^2))\langle 1 \rangle$$

where $\deg(X) = 2$.

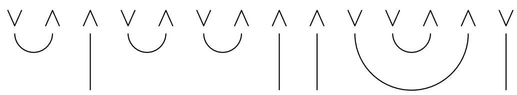
Now fix $V = \mathbb{C}(q)^2$ with basis $\{v_0, v_1\} = \{\vee, \wedge\}$. We can write down two bases for $V^{\otimes n}$

Fixed points basis/Standard basis

$$i_1 i_2 \cdots i_n = i_1 \otimes i_2 \otimes \cdots \otimes i_n \qquad i_j \in \{\vee, \wedge\}$$

Attracting cells basis/KL-basis denoted $\underline{i_1 i_2 \cdots i_n}$. The procedure for computing this basis is as follows

- take $i_1 i_2 \cdots i_n$, and write it as



  by connecting pairs $\vee\wedge$.

- Get a morphism $\phi : V^{\otimes k} \to V^{\otimes n}$.

- leave out the i_k which are connected by cups in $i_1 i_2 \cdots i_n$ to get $i_1 \widehat{i_2 \cdots} i_n$. Then take

$$\underline{i_1 i_2 \cdots i_n} := \phi(i_1 \widehat{i_2 \cdots} i_n) \in V^{\otimes n}$$

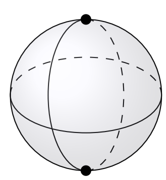
We say a *fixed point* (basis vector) v appears in (the closure of) an attracting cell (basis vector) b if v has non-trivial coefficient when b is expanded in the standard basis.

Remark. w appears in b if and only if w (a sequences of $\wedge$'s and $\vee$'s) and b (A cup diagram) is *oriented* with the fixed number of $\wedge$'s. In such a diagram, the degree equals the exponent of q.

Example. Let

$$X = \mathbb{P}^1(\mathbb{C}) = \{L \subset \mathbb{C}^2\} = SL(2, \mathbb{C}) / \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

which admits an action of $T = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$



If $\mathbb{C}^2 = \langle e_1, f_1 \rangle$, we get fixed points $\langle e_1 \rangle = L_1$ and $\langle f_1 \rangle = L_1$. For p a fixed point of T, we can consider

$$A_p := \overline{\left\{ x \in x \mid \lim_{t \to 0} T.x = p \right\}}$$

and compute:

$$T.[a, b] = [ta, t^{-1}b] = [t^2 a, b] \xrightarrow{t \to 0} [0, b] = [0, 1] = \langle f_1 \rangle$$

So, for $b \neq 0$,

$$\begin{aligned} A_{\langle f_1 \rangle} &= A_{[0,1]} = \mathbb{P}^1 \\ A_{\langle e_1 \rangle} &= A_{[1,0]} = \langle e_1 \rangle = [0, 1] \end{aligned}$$

Theorem (Khovanov, Brundan-S.)**.** *Fix n, fix $r := \#\wedge$'s. Consider the graded vector space*

$$B_{r,n} := \left\{ \underline{a}\lambda\underline{b} \mid \substack{a=a_1a_2\cdots a_n \\ b=b_1b_2\cdots b_n \\ \lambda=\lambda_1\lambda_2\cdots\lambda_n} \right\}$$

Such that $\overset{\lambda}{\underline{a}}, \overset{\lambda}{\underline{b}}$ oriented. Then there is a graded algebra structure on $B_{r,n}$ with primitive idempotents being the degree 0 elements $\underline{\lambda}\lambda\underline{\lambda}$.

Remark.

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